Renormalization Proof for Massive φ_4^4 Theory on Riemannian Manifolds

Christoph Kopper*
Centre de Physique Théorique, CNRS, UMR 7644
Ecole Polytechnique
F-91128 Palaiseau, France

Volkhard F. Müller†
Fachbereich Physik, Technische Universität Kaiserslautern
D-67653 Kaiserslautern, Germany

Abstract

In this paper we present an inductive renormalizability proof for massive φ_4^4 theory on Riemannian manifolds, based on the Wegner-Wilson flow equations of the Wilson renormalization group, adapted to perturbation theory. The proof goes in hand with bounds on the perturbative Schwinger functions which imply tree decay between their position arguments. An essential prerequisite are precise bounds on the short and long distance behaviour of the heat kernel on the manifold. With the aid of a regularity assumption (often taken for granted) we also show, that for suitable renormalization conditions the bare action takes the minimal form, that is to say, there appear the same counter terms as in flat space, apart from a logarithmically divergent one which is proportional to the scalar curvature.

* kopper@cpht.polytechnique.fr
† vfm@physik.uni-kl.de

1 Introduction

Among the different schemes deviced to prove the perturbative renormalizability of a local quantum field theory, the one based on the Wegner-Wilson differential flow equations of the Wilson renormalization group shows a distinctive characteristic: it circumvents completely the combinatoric complexity of generating Feynman diagrams and the subsequent cumbersome analysis of Feynman integrals with in general overlapping divergences. Initiated by Polchinski [Pol], this approach to renormalization has now been adapted to a wide variety of physically interesting instances. Partial reviews of the rigorous work which started from [KKS] may be found in [Kop1], [Sal], [Kop2], [Mü]. It is tempting to extend the approach via Wilson's flow equation further to prove the perturbative renormalizability of a quantum field theory defined on curved spacetime. There is a caveat, however. Using functional integration, one actually deals with a quantum field theory defined on a "Euclidean section" of curved spacetime, i.e. on a Riemannian manifold. In contrast to flat space there is no Wick rotation of Lorentzian curved spacetime, in general. Nevertheless, beyond static spacetimes, on particular nonstatic ones the analytic continuation of a quantum field theory to a corresponding Euclidean formulation has been rigorously shown recently: Bros, Epstein and Moschella [BEM] considered a quantum field theory on the anti-de Sitter (AdS) spacetime within a Wightman-type approach. As a consequence of certain spectral assumptions they show that the n-point correlation functions admit an analytic continuation to tuboidal domains (of n copies) of the complexified covering space of the AdS spacetime. This continuation includes the Euclidean AdS spacetime and satisfies there Osterwalder-Schrader positivity. Euclidean AdS spacetime is a Riemannian manifold with constant negative curvature.[1] Moreover, Birke and Fröhlich [BiFr], establishing in an algebraic approach Wick rotation of quantum field theories at finite temperature, also presented a reconstruction of quantum field theories on specific curved spacetimes from corresponding imaginary-time formulations, using group-theoretical techniques.

Our work starts straight considering a Riemannian manifold as given "spacetime". This manifold is assumed to be geodesically complete and to have all its sectional curvatures confined by a negative lower and a positive upper bound. We then study perturbative renormalizability of massive φ_4^4-theory defined on such a manifold by analysing the generating functional L^{Λ,Λ_0} of connected (free propagator) amputated Schwinger functions (CAS). From the physical point of view it seems justified to restrict to this class of manifolds, since in situations where curvature becomes large or where singularities appear the

[1] It is called 'hyperbolic space' in the mathematical literature.

treatment of gravity as a classical background effect becomes questionable anyway. As there is no translation symmetry, the CAS and thus the system of flow equations relating them have to be dealt with in position space. Establishing bounds involving these CAS we heavily rely on global lower and upper bounds for the heat kernel on the manifold, found in the mathematical literature.

Around the beginning of the eighties a considerable amount of work was carried out to formulate quantum field theory perturbatively on curved spacetime. Based on the intuition that ultraviolet divergences involve arbitrarily short wavelenghts, an approximating local momentum space representation of the Feynman propagator in curved spacetime was developed in [Bir],[BPP],[BuPn] for the ϕ^4-theory and generalized in [BuPr],[Bun1]. Combined with dimensional regularization, the euclideanized ϕ^4-theory was then shown to be renormalizable with local counterterms in one- and two-loop order. Furthermore, choosing the same general approach, Bunch [Bun2] has demonstrated the BPHZ renormalization of the ϕ^4-theory on euclideanized curved spacetime, by taking into account the power counting singular contributions in the asymptotic expansion of the propagator around its euclidean form. A different kind of generally applicable dimensional regularization scheme has been given by Lüscher [Lü], who applies it to the ϕ^4-theory on an arbitrary compact four-dimensional manifold with positive metric and to the Yang-Mills gauge theory on S^4. He also shows the renormalizability of the ϕ^4-theory by local counterterms at the one- and two-loop levels. Further references on work before 1982 can be found in the monograph [BiDa]. More recently, the perturbative construction of the ϕ^4-theory has been performed in an algebraic setting by Brunetti, Fredenhagen, Hollands and Wald [BrFr],[HoWa1],[HoWa2]. These authors adapted the renormalization method of Epstein and Glaser to construct the algebra of local, covariant quantum fields of the ϕ^4-model on a globally hyperbolic curved spacetime to any order of the perturbative expansion, making use of techniques from microlocal analysis. The crucial notion of a local and covariant quantum field introduced in [HoWa1,2] has been further formalized by Brunetti, Fredenhagen and Verch [BFV].

This paper is organized as follows : In Section 2 we collect and slightly adapt global bounds on the heat kernel found in the mathematical literature, which are pertinent to our treatment. The action considered and the system of perturbative flow equations satisfied by the CAS is set up in Section 3. To establish bounds on the CAS, being distributions, they have to be folded first with test functions. In Section 4 a suitable class of test functions is introduced, together with tree structures with the aid of which the bounds to be derived on the CAS are going to be expressed. In Section 5 we state the boundary and the renormalization conditions used to integrate the flow equations of the

irrelevant and relevant terms, respectively. The flow equations permit to be quite general in this respect, englobing basically all situations of interest. Section 6 is the central one of this paper. We state and prove inductive bounds on the Schwinger functions which, being uniform in the cutoff, directly lead to renormalizability. Beyond they imply tree decay of the Schwinger functions between their external points. The last section is devoted to the proof that the bare action of the theory may be chosen minimally, i.e. with position independent counter terms apart from one (logarithmically divergent) term which is proportional to the scalar curvature of the manifold. Here we have to make the assumption that geometric quantities on the manifold have a smooth expansion (to lowest orders) w.r.t. contributions of curvature terms of increasing mass dimension.

2 The heat kernel

We consider geodesically complete simply connected Riemannian manifolds $\mathcal{M}$ of dimension n without boundary, whose sectional curvatures are bounded between two constants $-k^2$ and κ^2. The related heat kernel then has the following properties :

$$K(t,x,y) \in C^\infty((0,\infty)\times\mathcal{M}\times\mathcal{M}) \ , \tag{1}$$

$$0 < \ K(t,x,y) < \infty \ , \tag{2}$$

$$K(t,x,y) = \ K(t,y,x) \ , \tag{3}$$

$$\int_{\mathcal{M}} K(t,x,y)\ dV(y) \ = \ 1 \ , \tag{4}$$

$$K(t_1+t_2,x,y) = \int_{\mathcal{M}} K(t_1,x,z)\ K(t_2,z,y)\ dV(z) \ . \tag{5}$$

Stochastic completeness (4) holds due to the assumed bounded curvature, cf. [Tay, ch.6, Prop.2.3]. Mathematicians have established quite sharp pointwise bounds on the respective heat kernels of various classes of manifolds. We are going to explicit now some of these bounds, because we will rely on them in the subsequent construction. Some bounds are known to hold for $0<t<T$, others for $0<t$ or $T<t$. We will write $t_\delta = t(1+\delta)$ where the parameter δ satisfies $0<\delta<1$ and may be chosen arbitrarily small. Furthermore $c,\ C,$ collectively denote constants which depend on δ, n and - if involved in the claim - on k^2, T.

On complete Riemannian manifolds of dimension n with nonnegative Ricci curvature the heat kernel satifies the lower and upper bounds [LiYa, Dav1]

$$\frac{c}{\sqrt{|\mathcal{B}(x,t^{1/2})|\,|\mathcal{B}(y,t^{1/2})|}}\, e^{-\frac{d^2(x,y)}{4t(1-\delta)}} \ \le\ K(t,x,y)\ \le\ \frac{C}{\sqrt{|\mathcal{B}(x,t^{1/2})|\,|\mathcal{B}(y,t^{1/2})|}}\, e^{-\frac{d^2(x,y)}{4t(1+\delta)}} \tag{6}$$

valid for all $x, y \in \mathcal{M}$ and $t > 0$.
In the case of negative Ricci curvature bounded below let $E \geq 0$ denote the bottom of the spectrum of the operator $-\Delta$. Then it holds, see [Dav2, Theorems 16 and 17] :
If $\delta > 0$, there exists a constant c_δ such that

$$K(t,x,y) \leq c_\delta \big(|\mathcal{B}(x,t^{1/2})|\, |\mathcal{B}(y,t^{1/2})| \big)^{-\frac{1}{2}} \exp\Big(-\frac{d^2(x,y)}{4t(1+\delta)}\Big) \tag{7}$$

for $0 < t < 1$ and for all $x, y \in \mathcal{M}$, whereas

$$K(t,x,y) \leq c_\delta \Big(|\mathcal{B}(x,1)|\, |\mathcal{B}(y,1)| \Big)^{-\frac{1}{2}} \exp\Big\{ (\delta - E)t - \frac{d^2(x,y)}{4t(1+\delta)} \Big\} \tag{8}$$

for $1 \leq t < \infty$ and for all $x, y \in \mathcal{M}$.
Moreover, a lower bound of the form appearing in (6) holds here, too, however restricted to $0 < t < T$, [Var].
In addition, given bounded sectional curvature $-k^2 \leq Sec_{\mathcal{M}} \leq \kappa^2$, there is the lower bound

$$K(t,x,y) \geq c \, \exp\Big(-\tilde{E}t - C\,\frac{d^2(x,y)}{t}\Big) \tag{9}$$

for all $x, y \in \mathcal{M}$ and for $t > T$, with constants $c, C > 0$ and $\tilde{E} > E$, possibly much larger, [Gri, ch. 7.5].
On a Cartan-Hadamard manifold of dimension n, i.e. a geodesically complete simply connected noncompact Riemannian manifold with nonpositive sectional curvature, and assuming that the sectional curvature is bounded below by $-K^2$, we also have for all $x, y \in \mathcal{M}$, and for $0 < t$, [Gri, ch. 7.4],

$$K(t,x,y) \;\leq\; \frac{C}{\min(1, t^{n/2})} \exp\Big(-\frac{(n-1)^2}{4} K^2 t - \frac{d^2(x,y)}{4t_\delta}\Big) . \tag{10}$$

For later use we extract from these bounds particular versions valid for four-dimensional complete Riemannian manifolds whose sectional curvatures may range between two constants $-k^2$ and κ^2.
From volume comparison, cf. e.g. [Cha, sect.3.4], follows:
i) If all sectional curvatures of $\mathcal{M}$ have values in $[-k^2, 0]$, $k > 0$, fixed, then

$$\frac{\pi^2}{2} t^2 \leq |\mathcal{B}(x,t^{1/2})| \leq \frac{\pi^2}{2} t^2 \, h_4(k\, t^{1/2}) \tag{11}$$

with the positive increasing function

$$h_4(r) = \frac{\cosh(3r) - 9\cosh r + 8}{3r^4}, \quad h_4(0) = 1\,,$$

ii) if all sectional curvatures of $\mathcal{M}$ have values in $[0, \kappa^2]$, $\kappa > 0$, fixed, then [2]

$$\frac{\pi^2}{2} t^2 \, s_4(\kappa \, t^{1/2}) \leq |\mathcal{B}(x, t^{1/2})| \leq \frac{\pi^2}{2} t^2, \quad \text{for} \quad \kappa \, t^{1/2} < \pi, \tag{12}$$

with the positive decreasing function, $0 < r < \pi$,

$$s_4(r) = \frac{\cos(3r) - 9\cos r + 8}{3r^4}, \quad s_4(0) = 1 \, .$$

Taking (6) together with (12), as well as taking (10) or (7) and the lower bound from (6) [3] together with (11), we obtain, restricting [4] to $0 < t < T$:

$$\frac{c}{t^2} \exp(-\frac{d^2(x,y)}{4t(1-\delta)}) \;\leq\; K(t,x,y) \;\leq\; \frac{C}{t^2} \exp(-\frac{d^2(x,y)}{4t(1+\delta)}) \, . \tag{13}$$

The constants c, C depend on k^2, κ^2, δ, T, but do not depend on t. As a consequence of this lower and upper bound we obtain under the same conditions

$$d^s(x,y) \, K(t,x,y) \;\leq\; c' \; t^{s/2} \; K(t_{\delta'}, x, y) \quad \text{for } t \leq T, \tag{14}$$

with $\delta' > \delta$. For $1 \leq s \leq 3$ we also need the bound

$$|\nabla^s K(t,x,y)| \;\leq\; C \; t^{-s/2} \; K(t_\delta, x, y) \tag{15}$$

based on [CLY], [Dav3] and valid for $0 < t < T$. Here ∇^s denotes a covariant derivative of order s w.r.t. x and the norm is that of (A.32). The constant C here also depends on the norm of the covariant derivatives of the curvature tensor up to order $s-1$. From (15) and the heat equation it follows directly that

$$|\partial_t K(t,x,y)| \;\leq\; C \; t^{-1} \; K(t_\delta, x, y) \, . \tag{16}$$

Finally we note the following recently proven bound on the logarithmic derivative of the heat kernel [SoZh] which holds for $Ric_{\mathcal{M}} \geq -k^2$ and for $0 < t < T$:

$$\frac{|\nabla K(t,x,y)|}{K(t,x,y)} \;\leq\; O(1) \; \frac{1}{t^{1/2}} \; (1 + \frac{d^2(x,y)}{t}) \, . \tag{17}$$

In closing this section we remark that the restriction to manifolds $\mathcal{M}$ of the kind considered is not dictated by the validity of our methods of proof. It rather seems to be a choice which is reasonable and interesting on physical grounds.

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[2] The restriction on t accounts for the injectivity radius of the manifold.

[3] Remember the statement after (8).

[4] The restriction is necessary both for the upper and lower bounds.

3 The Action and the Flow Equations

The regularized (free) propagator is given in terms of the heat kernel by

$$C^{\varepsilon,t}(x,y) \;=\; \int_{\varepsilon}^{t} dt'\; e^{-m^2\, t'}\;\; K(t',x,y)\;. \tag{18}$$

Its derivative w.r.t. t is denoted as

$$C_t(x,y) := \partial_t C^{\varepsilon,t}(x,y) = \; e^{-m^2\, t}\;\; K(t,x,y)\;. \tag{19}$$

We assume $0<\varepsilon\le t<\infty$ so that the flow parameter t takes the role of a long distance cutoff, whereas ε is a short distance regularization. The full propagator is recovered for $\varepsilon=0$ and $t\to\infty$. For finite ε and in finite volume the positivity and regularity properties of $C^{\varepsilon,t}$ permit to define the theory rigorously from the functional integral

$$e^{-\frac{1}{\hbar}(L^{\varepsilon,t}(\varphi)+I^{\varepsilon,t})} \;=\; \int d\mu_{\varepsilon,t}(\phi)\; e^{-\frac{1}{\hbar}L^{\varepsilon,\varepsilon}(\phi+\varphi)}\;, \quad L^{\varepsilon,t}(0):=0\;, \tag{20}$$

where the factors of $\hbar$ have been introduced to allow for a consistent loop expansion in the sequel. In (20) $d\mu_{\varepsilon,t}(\phi)$ denotes the Gaussian measure with covariance $\hbar C^{\varepsilon,t}(x,y)$. The test function φ here is supposed to be in the support of the Gaussian measures $d\mu_{t',t''}(\phi)$, $\varepsilon\le t'\le t''<\infty$, which implies in particular that it is in $C^\infty(\mathcal{M})$. The normalization factor $e^{-\frac{1}{\hbar}I^{\varepsilon,t}}$ is due to vacuum contributions. It diverges in infinite volume so that we can take the infinite volume limit only when it has been eliminated. We do not make the finite volume explicit here since it plays no role in the sequel.

The functional $L^{\varepsilon}(\varphi) \;:=\; L^{\varepsilon,\varepsilon}(\varphi)$ is the bare (inter)action including counterterms, viewed as a formal power series in $\hbar$. The superscript ε indicates the UV cutoff. For shortness we will pose in the following, with $x,y\in\mathcal{M}$, $\vec{x}=(x_1,\cdots,x_n)\in\mathcal{M}^{\times n}$,

$$\int_x \;:=\; \int_{\mathcal{M}} dV(x)\,, \qquad \int_{\vec{x}} \;:=\; \prod_{i=1}^{n}\int_{\mathcal{M}} dV(x_i)\,,$$

and

$$\tilde{\delta}(x,y) \;:=\; |g|^{-1/2}(x)\;\delta(x,y)\;.$$

As is known from lowest order calculations [Bir],[BPP],[Lü], in curved spacetime there will appear an additional counterterm of the type $\int_x R(x)\,\varphi^2(x)$ which is proportional to the scalar curvature $R(x)$ of the spacetime manifold $\mathcal{M}$ considered. So the bare interaction for the symmetric φ_4^4 theory would be

$$L^{\varepsilon}(\varphi) = \frac{\lambda}{4!}\int_x \varphi^4(x) \;+\; \frac{1}{2}\int_x \;\{(a^{\varepsilon}+\xi^{\varepsilon}R(x))\varphi^2(x) + b^{\varepsilon} g^{\mu\nu}(x)\;\partial_\mu\varphi(x)\cdot\partial_\nu\varphi(x) + \frac{2}{4!}\,c^{\varepsilon}\varphi^4(x)\} \tag{21}$$

where $\lambda > 0$ is the renormalized coupling, and the cutoff dependent parameters $a^\varepsilon,\ \xi^\varepsilon,\ b^\varepsilon,\ c^\varepsilon$ - which remain to be fixed and which are directly related to the mass, curvature, wave function, and coupling constant counterterms [5] - will fulfill

$$a^\varepsilon,\ \xi^\varepsilon,\ b^\varepsilon,\ c^\varepsilon = O(\hbar)\,. \tag{22}$$

It seems to us that there is no a priori reason to restrict to bare interactions of this form. In fact, since there is no translation invariance in curved space time, *all* counter terms and even the coupling λ itself may be position dependent. Quite generally the bare action is not a directly observable physical object, and the constraints on its form stem from the symmetry properties of the theory which are imposed, on its field content and on the form of the propagator. The symmetry properties depend in particular on the renormalization conditions which fix the physical (relevant) parameters of the theory. They might be position dependent : e.g. a local scattering experiment performed at different places at the same external momenta might give different cross sections and, as a consequence of this, the renormalized coupling, fixed in terms of the cross section, would be position dependent. It is therefore natural to admit more general bare interactions [6]

$$L^\varepsilon(\varphi) = \int_x \frac{\lambda(x)}{4!}\,\varphi^4(x) +$$

$$\frac{1}{2}\int_x \{a^\varepsilon(x)\,\varphi^2(x) + u^{\mu,\varepsilon}(x)\,\varphi(x)\,\nabla_\mu\varphi(x) + \hat{b}^\varepsilon(x) g^{\mu\nu}(x)\ \partial_\mu\varphi(x)\cdot\partial_\nu\varphi(x) + \frac{2}{4!}\,c^\varepsilon(x)\varphi^4(x)\}\,.$$

Here $\lambda(x),\ a^\varepsilon(x),\ \hat{b}^\varepsilon(x),\ c^\varepsilon(x)$ are general scalars and $u^{\mu,\varepsilon}(x)$ is a general vector, all functions are supposed to be smooth, and $|\lambda(x)|$ (of course) uniformly bounded on $\mathcal{M}$. When calculating the two point function $\mathcal{L}^\varepsilon(x_1,x_2) = \ \delta/\delta_{\varphi(x_1)}\,\delta/\delta_{\varphi(x_2)}\,L^\varepsilon(\varphi)|_{\varphi\equiv 0}$ from this bare action one obtains

$$\begin{aligned}\mathcal{L}^\epsilon_2(x_1,x_2) &= a^\epsilon(x_1)\,\tilde{\delta}(x_2,x_1) - \frac{1}{2}(\nabla_\mu u^{\mu,\epsilon})(x_1)\,\tilde{\delta}(x_2,x_1)\\ &- |g(x_2)|^{-\frac{1}{2}}\,\partial^{(2)}_\mu\,\hat{b}^\varepsilon(x_2)\,g^{\mu\nu}(x_2)\,|g(x_2)|^{\frac{1}{2}}\,\partial^{(2)}_\nu\,\tilde{\delta}(x_2,x_1)\ .\end{aligned} \tag{23}$$

This means that $u^{\mu,\epsilon}$ only contributes via its divergence, and that the contribution $\sim\ \nabla_\mu u^{\mu,\epsilon}$ can be absorbed in a^ϵ. On the other hand the particular tensor coupling $\hat{b}^\varepsilon(x)g^{\mu\nu}(x)$ can be generalized - without changing symmetry properties - into a smooth

[5] Since it is not necessary in the flow equation framework to introduce bare fields in distinction from renormalized ones (our field is the renormalized one in this language), there is a slight difference, which is to be kept in mind only when comparing to other schemes.

[6] One could of course be even more general.

symmetric $(2,0)$-tensor field $b^{\mu\nu,\varepsilon}(x)$. In (23) the product $\hat{b}^{\varepsilon}(x) g^{\mu\nu}(x)$ is then replaced by $b^{\mu\nu,\varepsilon}(x)$, and we recognise that in (23) the generalisation

$$\Delta^{(b)} := |g(x)|^{-\frac{1}{2}} \partial_\mu \, b^{\mu\nu}(x) \, |g(x)|^{\frac{1}{2}} \partial_\nu \tag{24}$$

of the Laplace-Beltrami operator Δ, (A.4), appears. We thus adopt a bare interaction of the form

$$L^{\varepsilon}(\varphi) = \int_x \frac{\lambda(x)}{4!} \varphi^4(x) + \frac{1}{2} \int_x \{ \tilde{a}^{\varepsilon}(x) \, \varphi^2(x) + b^{\mu\nu,\varepsilon}(x) \, \partial_\mu \varphi(x) \cdot \partial_\nu \varphi(x) + \frac{2}{4!} c^{\varepsilon}(x) \varphi^4(x) \} \tag{25}$$

with smooth scalar functions $\tilde{a}^{\varepsilon}(x)$, $c^{\varepsilon}(x)$ and a smooth symmetric tensor field $b^{\mu\nu,\varepsilon}(x)$ - which remain to be fixed and which are of (at least) order $\hbar$.

The flow equation (FE) is obtained from (20) on differentiating w.r.t. t. It is a differential equation for the functional $L^{\varepsilon,t}$:

$$\partial_t (L^{\varepsilon,t} + I^{\varepsilon,t}) = \frac{\hbar}{2} \langle \frac{\delta}{\delta\varphi}, C_t \frac{\delta}{\delta\varphi} \rangle L^{\varepsilon,t} - \frac{1}{2} \langle \frac{\delta}{\delta\varphi} L^{\varepsilon,t}, C_t \frac{\delta}{\delta\varphi} L^{\varepsilon,t} \rangle \ . \tag{26}$$

By $\langle \ , \ \rangle$ we denote the standard inner product in $L^2(\mathcal{M}, dV(x))$. The FE can also be stated in integrated form

$$e^{-\frac{1}{\hbar}(L^{\varepsilon,t}(\varphi) + I^{\varepsilon,t})} = e^{\hbar \Delta_{\mathcal{F}}(\varepsilon,t)} \, e^{-\frac{1}{\hbar} L^{\varepsilon,\varepsilon}(\varphi)} \ . \tag{27}$$

The functional Laplace operator $\Delta_{\mathcal{F}}(\varepsilon,t)$ is given by

$$\Delta_{\mathcal{F}}(\varepsilon,t) = \frac{1}{2} \langle \delta_\varphi, \ C^{\varepsilon,t} \ \delta_\varphi \rangle$$

using the notation $\delta_{\varphi(x)} = \delta/\delta\varphi(x)$. We may expand $L^{\varepsilon,t}(\varphi)$ w.r.t. the number of fields φ setting

$$L_n^{\varepsilon,t}(\varphi) := \frac{1}{n!} \frac{\partial^n}{\partial \kappa^n} L^{\varepsilon,t}(\kappa\varphi)|_{\kappa=0} \ . \tag{28}$$

The functional $L^{\varepsilon,t}(\varphi)$ can also be expanded in a formal powers series w.r.t. $\hbar$, and in a double series w.r.t. $\hbar$ and the number of fields

$$L^{\varepsilon,t}(\varphi) = \sum_{l=0}^{\infty} \hbar^l \, L_l^{\varepsilon,t}(\varphi) = \sum_{l=0}^{\infty} \hbar^l \sum_{n=2}^{\infty} L_{n,l}^{\varepsilon,t}(\varphi) \ , \quad L_{2,0}^{\varepsilon,t}(\varphi) \equiv 0 \ . \tag{29}$$

Corresponding expansions for $\tilde{a}^{\varepsilon}(x)$, $b^{\mu\nu,\varepsilon}(x)$, $c^{\varepsilon}(x)$ are $\tilde{a}^{\varepsilon}(x) = \sum_{l\geq 1} \tilde{a}_l^{\varepsilon}(x) \hbar^l$ etc. We can then rewrite (26) in loop order l as

$$\partial_t L_{n,l}^{\varepsilon,t}(\varphi) =$$

$$\frac{1}{2}\int_{x,y} C_t(x,y)\,\Big[\delta_{\varphi(x)}\;\delta_{\varphi(y)}\,L^{\varepsilon,t}_{n+2,l-1}(\varphi)\;-\sum_{\substack{n_1+n_2=n+2\\ l_1+l_2=l}}(\delta_{\varphi(x)}\,L^{\varepsilon,t}_{n_1,l_1}(\varphi))\;\delta_{\varphi(y)}\,L^{\varepsilon,t}_{n_2,l_2}(\varphi)\Big]\;. \quad (30)$$

From $L^{\varepsilon,t}_l(\varphi)$ we obtain the connected amputated Schwinger functions of loop order l as

$$\mathcal{L}^{\varepsilon,t}_{n,l}(x_1,\ldots,x_n)\;:=\;\delta_{\varphi(x_1)}\ldots\delta_{\varphi(x_n)}L^{\varepsilon,t}_l|_{\varphi\equiv 0}\;. \quad (31)$$

It is straightforward to realize that the $\mathcal{L}^{\varepsilon,t}_{n,l}$ are distributions
1) which are completely *symmetric* w.r.t. permutations of the arguments $(x_1,\ldots,x_n)$
2) and which *fall off rapidly* with the distances $d(x_i,x_j)$.
These facts follow from (25), (27) and the properties of the regularized propagator (6), (13). The distributional character of the $\mathcal{L}^{\varepsilon,t}_{n,l}$ is related to the fact that we consider *amputated* Schwinger functions. Thus there is associated a factor of $\tilde{\delta}(x_i,z)$ to the external line joining x_i to an internal vertex at z which is integrated over. Therefore the distributional character is different according to whether one or several (up to three) external lines end in a given z-vertex. From the point of view of the FE the distributional character is a consequence of the boundary conditions, see (25) and (60), (61). Note that by (1) the propagators $C^{\varepsilon,t}$ which join different vertices are smooth functions of their position arguments for $\varepsilon>0$. The two-point function is the most divergent object as regards its flow for small t. But the distributional structure of its regularized version is particularly simple. In fact it follows from (25) and (27) that $\mathcal{L}^{\varepsilon,t}_{2,l}(x_1,x_2)$ can be written as a sum over regularized Feynman amplitudes with vertices from L^{ε} and with regularized propagators which satisfy (1), (6). Thus the only distributional singularities appearing are of the form $\tilde{\delta}(x_2,x_1)$ and $\Delta^{(b^{\varepsilon,\varepsilon}_l)}{}_{x_2}\;\tilde{\delta}(x_2,x_1)$. Thus $\mathcal{L}^{\varepsilon,t}_{2,l}(x_1,x_2)$ can be written as a linear combination of these two contributions and a smooth function $f(x_1,x_2)$ of rapid decrease in $d(x_1,x_2)$.

The FE for the Schwinger functions derived from (30) takes the following form :

$$\partial_t\mathcal{L}^{\varepsilon,t}_{n,l}(x_1,\ldots,x_n)=\frac{1}{2}\int_{x,y} C_t(x,y)\,\Big\{\mathcal{L}^{\varepsilon,t}_{n+2,l-1}(x_1,\ldots,x_n,x,y)\;- \quad (32)$$

$$\sum_{\substack{l_1+l_2=l,\\ n_1+n_2=n}}\Big[\mathcal{L}^{\varepsilon,t}_{n_1+1,l_1}(x_1,\ldots,x_{n_1},x)\;\mathcal{L}^{\varepsilon,t}_{n_2+1,l_2}(y,x_{n_1+1},\ldots,x_n)\Big]_{sym}\Big\}\,.$$

Here sym means symmetrization - i.e. summing over all permutations [7] of $(x_1,\ldots,x_n)$ *modulo those* which only rearrange the arguments of a factor. [8]

[7] by our choice of normalization there is no normalization factor to be divided out

[8] This may be implemented by counting only the configurations in which the permuted position variables appearing in $\mathcal{L}^{\varepsilon,t}_{n_1+1}$ and $\mathcal{L}^{\varepsilon,t}_{n_2+1}$ appear in lexicographic order.

For the renormalization proof we also need the FE for the Schwinger functions derived w.r.t. the UV cutoff ε. Integrating the FE over t' between ε and t and then deriving w.r.t. ε we obtain

$$\begin{aligned}\partial_\varepsilon \mathcal{L}^{\varepsilon,t}_{n,l}(x_1,\ldots,x_n) = \; & \partial_\varepsilon \mathcal{L}^{\varepsilon,\varepsilon}_{n,l}(x_1,\ldots,x_n) - \frac{1}{2}\int_{x,y} C_\varepsilon(x,y) \Big\{ \mathcal{L}^{\varepsilon,\varepsilon}_{n+2,l-1}(x_1,\ldots,x_n,x,y) - \\ & \sum_{\substack{l_1+l_2=l,\\ n_1+n_2=n}} \Big[\mathcal{L}^{\varepsilon,\varepsilon}_{n_1+1,l_1}(x_1,\ldots,x_{n_1},x)\; \mathcal{L}^{\varepsilon,\varepsilon}_{n_2+1,l_2}(y,x_{n_1+1},\ldots,x_n)\Big]_{sym} \Big\} + \qquad (33)\\ & \frac{1}{2}\int_{x,y}\int_\varepsilon^t dt'\; C_{t'}(x,y) \Big\{ \partial_\varepsilon \mathcal{L}^{\varepsilon,t'}_{n+2,l-1}(x_1,\ldots,x_n,x,y) - \\ & \sum_{\substack{l_1+l_2=l,\\ n_1+n_2=n}} \Big[\partial_\varepsilon \Big(\mathcal{L}^{\varepsilon,t'}_{n_1+1,l_1}(x_1,\ldots,x_{n_1},x)\; \mathcal{L}^{\varepsilon,t'}_{n_2+1,l_2}(y,x_{n_1+1},\ldots,x_n)\Big)\Big]_{sym} \Big\}\,.\end{aligned}$$

Integrating the FE instead from $t<1$ to $t=1$ and then deriving w.r.t. ε we get

$$\begin{aligned}\partial_\varepsilon \mathcal{L}^{\varepsilon,t}_{n,l}(x_1,\ldots,x_n) = \; & \partial_\varepsilon \mathcal{L}^{\varepsilon,1}_{n,l}(x_1,\ldots,x_n) - \qquad (34)\\ & \frac{1}{2}\int_{x,y}\int_t^1 dt'\; C_{t'}(x,y) \Big\{ \partial_\varepsilon \mathcal{L}^{\varepsilon,t'}_{n+2,l-1}(x_1,\ldots,x_n,x,y) - \\ & \sum_{\substack{l_1+l_2=l,\\ n_1+n_2=n}} \Big[\partial_\varepsilon \Big(\mathcal{L}^{\varepsilon,t'}_{n_1+1,l_1}(x_1,\ldots,x_{n_1},x)\; \mathcal{L}^{\varepsilon,t'}_{n_2+1,l_2}(y,x_{n_1+1},\ldots,x_n)\Big)\Big]_{sym} \Big\}\,.\end{aligned}$$

4 Test functions and Tree structures

The distributional character of the $\mathcal{L}^{\varepsilon,t}_{n,l}$ necessitates the introduction of test functions against which they will be integrated. Later on we will only use a subclass of the test functions introduced in the subsequent definition, see (42).

Definition 1 : For $n \in \mathbb{N}$ we set

$$\mathcal{H}_n := \{\varphi(\vec{x}_n) = \varphi_1(x_1)\ldots\varphi_n(x_n) \mid \varphi_i \in C^\infty(\mathcal{M}) \cap L^\infty(\mathcal{M})\}\,.$$

We wrote $\vec{x}_n = (x_1,\ldots,x_n)$ and we shall write [9] $x_{2,n} = (x_2,\ldots,x_n) \in \mathcal{M}^{\times(n-1)}$.
For $\varphi \in \mathcal{H}_{n-1}$ we set

$$\mathcal{L}^{\varepsilon,t}_{n,l}(x_1,\varphi) := \int_{x_{2,n}} \mathcal{L}^{\varepsilon,t}_{n,l}(\vec{x}_n)\; \varphi(x_{2,n})\,. \qquad (35)$$

[9]By the bosonic symmetry of the $\mathcal{L}^{\varepsilon,t}_{n,l}$ all bounds are independent of the particular role assigned to the coordinate x_1, which can be exchanged with any other coordinate.

The regularized Schwinger functions are obviously linear w.r.t. the test functions:

$$\mathcal{L}^{\varepsilon,t}_{n,l}(x_1, a\,\varphi_1 + b\,\varphi_2) = \; a\; \mathcal{L}^{\varepsilon,t}_{n,l}(x_1,\varphi_1) \;+\; b\; \mathcal{L}^{\varepsilon,t}_{n,l}(x_1,\varphi_2)\;, \quad a,b \in \mathbb{C}\;, \tag{36}$$

and it also follows from from (25) and (27) and the properties of the regularized propagator (1), (10) that they satisfy

$$\mathcal{L}^{\varepsilon,t}_{n,l}(x_1,\varphi) \in \mathcal{H}_1\;.$$

We will also consider Schwinger functions multiplied by products of factors $\sigma(x_j,x_1)^\mu$, (A.21).

Definition 2 : We introduce a smooth (external) covector field $\omega_\mu(x)$ and form the bi-scalar insertions

$$E_{(i)} \equiv E(x_i,x_1\;;\omega) := \sigma(x_i,x_1)^\mu\,\omega_\mu(x_1), \qquad i = 2,\cdots,n\;, \tag{37}$$

and, more generally, for $r \in \mathbb{N}$,

$$E^{(r)}_{(i)} \equiv E(x_i,x_1\;;\omega^{(r)}) := \sigma(x_i,x_1)^{\mu_1}\ldots\sigma(x_i,x_1)^{\mu_r}\,\omega^{(r)}_{\mu_1\ldots\mu_r}(x_1) \tag{38}$$

with a smooth (external) symmetric covariant tensor field $\omega^{(r)}_{\mu_1\ldots\mu_r}(x)$ of rank r. We have, because of (A.22),

$$|E(x_i,x_1\;;\omega^{(r)})| \le |\,\omega^{(r)}(x_1)|\,d^{\,r}(x_i,x_1) \tag{39}$$

with the norm $|\,\omega^{(r)}(x_1)|$ according to (A.32). For $r \in \mathbb{N}$ we then pose

$$\mathcal{L}^{\varepsilon,t}_{n,l}(x_1,E^{(r)}_{(i)}\,\varphi) := \int_{x_{2,n}} E(x_i,x_1\;;\omega^{(r)})\,\mathcal{L}^{\varepsilon,t}_{n,l}(\vec{x}_n)\;\varphi(x_{2,n})\;. \tag{40}$$

Mostly we will suppress ω in the notation as we did in (40). Furthermore, for given $x_1,x_2 \in \mathcal{M}$ we consider the products

$$F^{(r)}_{(12)}\mathcal{L}^{\varepsilon,t}_{n,l}(x_1,x_2,\varphi) := d^{\,3-r}(x_1,x_2)E(x_2,x_1\;;\omega^{(r)}) \int_{x_{3,n}} \mathcal{L}^{\varepsilon,t}_{n,l}(\vec{x}_n)\;\varphi(x_{3,n}) \tag{41}$$

for $r = 0,1,2$, with $E \equiv 1$ if $r = 0$, and with $\varphi(x_{3,n}) \equiv 1$ (and no integration) if $n < 3$.

Definition 3 : i) A graph $G(V,\mathcal{P})$ is defined as a set of vertices V and a set $\mathcal{P}$ of unordered pairs p of vertices called lines/edges. Two lines are connected if they share a vertex in common. A graph is connected if for each pair of vertices (i,j) there exists a path of connected lines connecting i to j. A tree is a connected graph $G(V,\mathcal{P})$ with $|\mathcal{P}| = |V| - 1$. For a tree one can prove that the path of connected lines connecting i to j is *unique.* A rooted tree is a tree where one vertex in V has been chosen to be its root. The incidence number c_i of the vertex i in a tree is the number of distinct lines

containing i. The subset $V_e \subset V$, containing the vertices i with $c_i = 1$, excluding the root (if it has $c = 1$), is called the set of external vertices. All other vertices are called internal vertices. We denote by $\mathcal{T}^s$ the set of all trees such that $|V_e| = s-1$, $s \geq 2$. Subsequently we will consider trees where the set of vertices is identified[10] with a set of points in the manifold $\mathcal{M}$. For a tree $T^s \in \mathcal{T}^s$ we will call $x_1 \in \mathcal{M}$ its root vertex, and $Y = \{y_2, \ldots, y_s\}$ the set of points in $\mathcal{M}$ to be identified with its external vertices. Likewise we call $Z = \{z_1, \ldots, z_r\}$ with $r \geq 0$ the set of internal vertices of T^s.

ii) For $y_i \in Y$ there exists exactly one $p \in \mathcal{P}$ such that $y_i \in p$. For x_1 there exist $p_1, \ldots, p_{c_1} \in \mathcal{P}$ with $1 \leq c_1 \leq s-1$ such that $x_1 \in p_1, \ldots, x_1 \in p_{c_1}$. For $z_j \in Z$ there exist $p_1^{(z_j)}, \ldots, p_{c_j}^{(z_j)} \in \mathcal{E}$ with $2 \leq c_j \leq s$ such that $z_j \in p_1^{(z_j)}, \ldots, z_j \in p_{c_j}^{(z_j)}$. We call $c_1 = c(x_1)$ the incidence number of the root vertex and $c(z_j)$ the incidence number of the internal vertex z_j of the tree.

We call a line $p \in \mathcal{P}$ an external line of the tree if there exists y_i such that $y_i \in p$. The set of external lines is denoted $\mathcal{J}$. The remaining lines are called internal lines of the tree and are denoted by $\mathcal{I}$, hence $\mathcal{P} = \mathcal{J} \cup \mathcal{I}$.

iii) Denoting by v_c the number of vertices having incidence number c, it follows from the definition that $\sum_{c \geq 2}(c-2)\, v_c = s - 3 + \delta_{c_1,1}$. By T_l^s we denote a tree $T^s \in \mathcal{T}^s$ satisfying $v_2 + \delta_{c_1,1} \leq 3l - 2 + s/2$ for $l \geq 1$ and satisfying $v_2 = 0$ for $l = 0$. Then $\mathcal{T}_l^s$ denotes the set of all trees T_l^s. We indicate the external vertices and internal vertices of the tree by writing $T_l^s(x_1, y_{2,s}, \vec{z})$ with $y_{2,s} = (y_2, \ldots, y_s)$, $\vec{z} = (z_1, \ldots, z_r)$.

iv) We also define for $i \leq s$ the set of *twice rooted trees* denoted as $\mathcal{T}_l^{s,(12)}$. The trees $T_l^{s,(12)} \in \mathcal{T}_l^{s,(12)}$ are defined exactly as the trees T_l^s apart from the fact that they have two root vertices x_1, x_2 with the properties of ii) above, and $s-2$ external vertices.

Definition 4 : For a tree $T_l^{s+2}(x_1, y_{2,s+2}, \vec{z})$ we define the *reduced tree* $T_{l,y_i y_j}^s(x_1, y_2, \ldots, y_{i-1}, y_{i+1}, \ldots, y_{j-1}, y_{j+1}, \ldots, y_{s+2}, \vec{z}_{ij})$ to be the unique tree to be obtained from $T_l^{s+2}(x_1, y_{2,s+2}, \vec{z})$ through the following procedure :

i) By taking off the two external vertices y_i, y_j together with the external lines attached to them.

ii) By taking off the internal vertices - if any - which have acquired incidence number $c = 1$ through the previous process, and by also taking off the lines attached to them.

iii) If the process ii) has produced a new vertex of incidence number 1 go back to ii).

In the sequel we shall bound the CAS folded with test functions. Here we restrict to test

[10] In mathematically straight notation a vertex should be viewed as the image of an element of a discrete set under a mapping from this set into $\mathcal{M}$.

functions of the following form : Let $1 \leq s \leq n$ and $\tau = \tau_{2,s} = (\tau_2, \ldots, \tau_s)$ with $0 < \tau_i$

$$\varphi_{\tau,y_{2,s}}(x_{2,n}) \;:=\; \prod_{i=2}^{s} K(\tau_i, x_i, y_i) \;\prod_{i=s+1}^{n} \mathbf{1}(x_i)\ . \tag{42}$$

Here $\mathbf{1}(x) = 1 \;\; \forall x \in \mathcal{M}$. These test functions are factorized[11]. The nonconstant functions are smooth, globally defined and rapidly decreasing on $\mathcal{M}$. The pair τ_j, y_j determines the width and the center of localisation of the test function. This definition can be generalized by choosing any other subset of s coordinates among $x_2, \ldots, x_n$. We also define [12] for $2 \leq j \leq s$

$$\varphi^{(j)}_{\tau,y_{2,s}}(x_{2,n}) \;:=\; K^{(1)}(\tau_j, x_j, x_1; y_j) \prod_{i=2, i\neq j}^{s} K(\tau_i, x_i, y_i) \prod_{i=s+1}^{n} \mathbf{1}(x_i) \tag{43}$$

with

$$K^{(1)}(\tau_j, x_j, x_1; y_j) = \; K(\tau_j, x_j, y_j) - \; K(\tau_j, x_1, y_j)\ . \tag{44}$$

Definition 5: Given τ, $y_{2,s}$, $\delta > 0$, and a set of internal vertices $\vec{z} = (z_1, \ldots, z_r)$, and attributing positive parameters $t_{\mathcal{I}} = \{t_I | I \in \mathcal{I}\}$ to the internal lines, the *weight factor* $\mathcal{F}(t_{\mathcal{I}}, \tau; T^s_l(x_1, y_{2,s}, \vec{z}))$ *of a tree* $T^s_l(x_1, y_{2,s}, \vec{z})$ *at scales* $t_{\mathcal{I}}$ is defined as a product of heat kernels associated with the internal and external lines of the tree. We set

$$\mathcal{F}(t_{\mathcal{I}}, \tau; T^s_l(x_1, y_{2,s}, \vec{z})) := \prod_{I\in\mathcal{I}} C_{t_{I,\delta}}(I) \prod_{J\in\mathcal{J}} K(\tau_{J,\delta}, J)\ . \tag{45}$$

Here we denote by τ_J the entry τ_i in τ carrying the index of the external coordinate y_i in which the external line J ends. For $I = \{a, b\}$ the notation $C_{t_I}(I)$ stands for $C_{t_I}(a, b)$. We then also define the *integrated weight factor* of a tree by

$$\mathcal{F}(t, \tau; T^s_l\ ; x_1, y_{2,s}) \;:=\; \sup_{\{t_I | I\in\mathcal{I}, \varepsilon\leq t_I\leq t\}} \int_{\vec{z}} \mathcal{F}(t_{\mathcal{I}}, \tau; T^s_l(x_1, y_{2,s}, \vec{z}))\ . \tag{46}$$

It depends on ε, but note that its limit for $\varepsilon \to 0$ exists, and that typically the sup is expected to be taken for the maximal values of t admitted. Therefore we suppress the dependence on ε in the notation. Finally we introduce the shorthand notation for the *global weight factor* $\mathcal{F}_{s,l}(t, \tau; x_1, y_{2,s})$ *or more shortly* $\mathcal{F}_{s,l}(t, \tau)$ which is defined as follows

$$\mathcal{F}_{s,l}(t, \tau) \;\equiv\; \mathcal{F}_{s,l}(t, \tau; x_1, y_{2,s}) \;:=\; \sum_{T^s_l \in \mathcal{T}^s_l} \mathcal{F}(t, \tau; T^s_l\ ; x_1, y_{2,s})\ . \tag{47}$$

[11] The function $\mathbf{1}(x)$ is obtained on integrating $K(\tau, x, y)$ over y. This could be used to unify the notation.

[12] Note that $\varphi^{(j)}_{\tau,y_{2,s}}$ depends on x_1 which is not indicated.

In complete analogy we define the weight factors and global weight factors for twice rooted trees which we denote as $\mathcal{F}(t,\tau;T_l^{s,(12)}\ ;x_1,x_2,y_{3,s})$ resp. $\mathcal{F}_{s,l}^{(12)}(t,\tau;x_1,x_2,y_{3,s})$ or $\mathcal{F}_{s,l}^{(12)}(t,\tau)$. Following the definitions (45)-(47) we also define for $t\geq 1$

$$\mathcal{F}^{\,t}(\tau;T_l^s(x_1,y_{2,s},\vec{z})) := \sup_{\{t_I|I\in\mathcal{I},\,\varepsilon\leq t_I\leq 1\}} \prod_{I\in\mathcal{I}}[(C_{t_{I,\delta}}(I)+\int_1^t C_{t'}(I)\,dt')]\prod_{J\in\mathcal{J}} K(\tau_{J,\delta},J)\ , \quad (48)$$

$$\mathcal{F}^{\,t}(\tau;T_l^s\ ;x_1,y_{2,s})\ :=\ \int_{\vec{z}}\mathcal{F}^{\,t}(\tau;T_l^s(x_1,y_{2,s},\vec{z}))\ , \quad (49)$$

and

$$\mathcal{F}_{s,l}^{t}(\tau)\ :=\ \sum_{T_l^s\in\mathcal{T}_l^s}\mathcal{F}^{\,t}(\tau;T_l^s\ ;x_1,y_{2,s})\ . \quad (50)$$

For $s=1$ we set $\mathcal{F}_{1,l}(t,\tau)\ \equiv\ 1$.

We give more explicitly the form of $\mathcal{F}_{2,l}(t,\tau;x,y)$. It is by definition given through

$$\mathcal{F}_{2,l}(t,\tau;x,y)\ =\ \sum_{T_l^2}\mathcal{F}_{2,l}(t,\tau;T_l^2\ ;x,y)\ =\ K(\tau_\delta,x,y)\ +$$

$$\sum_{n=1}^{3l-2}\ \sup_{\{t_{I_i}|\varepsilon\leq t_{I_i}\leq t,\,i=1.\cdots,n\}}\ [\prod_{1\leq i\leq n}\int_{z_i}]\ C_{t_{I_1,\delta}}(x,z_1)\ldots C_{t_{I_n,\delta}}(z_{n-1},z_n)\ K(\tau_\delta,z_n,y)\ .$$

Using (5) we get

$$\mathcal{F}_{2,l}(t,\tau;x,y)\ =\sum_{n=0}^{3l-2}\ \sup_{\{t_{I_i}|\varepsilon\leq t_{I_i}\leq t,\,i=1.\cdots,n\}} C_{\tau_\delta+\sum_1^n t_{I_i,\delta}}(x,y)\ e^{m^2\tau_\delta}\ . \quad (51)$$

Let us shortly comment on why we are led to introduce tree structures and weight factors in our context. In fact we will establish bounds for the CAS (35) inductively by concluding from the CAS appearing on the r.h.s. of the FE (32) on the CAS appearing on the l.h.s. Assume we have bounds in terms of weight factors of trees for the $\mathcal{L}$'s on the r.h.s. The second contribution on the r.h.s. of (32) then lends itself immediately to reproduce such a bound if the factor, associated in our bound with a line of the tree, is a bound on $C_t(x,y)$, and if the vertices $x,\,y$ appearing in the bound are integrated over. This is the case for our definition of weight factors since in particular internal vertices are integrated over. For the first contribution on the r.h.s. of (32) we would like to pass from a tree associated with $\mathcal{L}_{n+2}$ to a tree associated with $\mathcal{L}_n$. This requires the bound to be such that its integration over $x,\,y$ against the factor $C_t(x,y)$ finally leads to an

expression bounded by a tree bound on $\mathcal{L}_n$. This is at the origin of the notion of reduced tree introduced above, where two external points have disappeared.

For simplicity we choose the test functions appearing in the weight factors to be heat kernels themselves. These form a sufficiently large set. However, to get inductive control of the local counter terms, we also have to admit the situation where some of the external coordinates are just integrated over all of $\mathcal{M}$. This leads to the general form of the admitted test functions (42).

5 Boundary and renormalization conditions

From the mathematical point of view the renormalization problem in the FE framework appears as a mixed boundary value problem. The relevant terms are fixed by renormalization conditions at a large value t_R of the flow parameter t, all other boundary terms are fixed at the short-distance cutoff $t = \varepsilon$.

To extract the relevant terms - contained in $\mathcal{L}^{\varepsilon,t}_{2,l}(x_1,\varphi)$ and $\mathcal{L}^{\varepsilon,t}_{l,4}(x_1,\varphi)$ - a covariant Taylor expansion with remainder term (A.28), (A.29) of the test function φ is used, $\varepsilon \le t$:

$$\mathcal{L}^{\varepsilon,t}_{2,l}(x_1,\varphi) \;=\; a^{\varepsilon,t}_l(x_1)\,\varphi(x_1) - f^{\mu,\varepsilon,t}_l(x_1)\,(\nabla_\mu\varphi)(x_1) - b^{\mu\nu,\varepsilon,t}_l(x_1)(\nabla_\mu\nabla_\nu\varphi)(x_1) + \ell^{\varepsilon,t}_{2,l}\,(x_1,\varphi)\,, \tag{52}$$

$$\mathcal{L}^{\varepsilon,t}_{4,l}(x_1,\varphi) = \; c^{\varepsilon,t}_l(x_1)\;\varphi_2(x_1)\;\varphi_3(x_1)\varphi_4(x_1) + \;\ell^{\varepsilon,t}_{4,l}(x_1,\varphi)\,. \tag{53}$$

Then the relevant terms appear as

$$a^{\varepsilon,t}_l(x_1) = \int_{x_2} \mathcal{L}^{\varepsilon,t}_{2,l}(x_1,x_2)\;, \quad f^{\mu,\varepsilon,t}_l(x_1) = \int_{x_2} \sigma(x_2,x_1)^\mu\;\mathcal{L}^{\varepsilon,t}_{2,l}(x_1,x_2)\;,$$

$$b^{\mu\nu,\varepsilon,t}_l(x_1) = -\,\frac{1}{2}\int_{x_2} \sigma(x_2,x_1)^\mu\;\sigma(x_2,x_1)^\nu\mathcal{L}^{\varepsilon,t}_{2,l}(x_1,x_2)\,, \tag{54}$$

$$c^{\varepsilon,t}_l(x_1) = \int_{x_2,x_3,x_4} \mathcal{L}^{\varepsilon,t}_{4,l}(x_1,\ldots,x_4)\,, \tag{55}$$

and the 'remainders' $\ell^{\varepsilon,t}_{2,l}$ and $\ell^{\varepsilon,t}_{4,l}$ have the respective forms

$$\ell^{\varepsilon,t}_{2,l}(x_1,\varphi) \;=\; \int_{x_2} \mathcal{L}^{\varepsilon,t}_{2,l}(x_1,x_2)\int_0^s dr\,\frac{(s-r)^2}{2!}\;\dot{x}^{\nu_3}_{12}(r)\,\dot{x}^{\nu_2}_{12}(r)\,\dot{x}^{\nu_1}_{12}(r)\big(\nabla_{\nu_3}\nabla_{\nu_2}\nabla_{\nu_1}\varphi\big)(x_{12}(r)) \tag{56}$$

where $s = d(x_1,x_2)$ and $x_{12}(r)$ is the point on the geodesic segment from x_1 to x_2 at arc length r; and

$$\ell^{\varepsilon,t}_{4,l}(x_1,\varphi) \;=\; \int_{x_2,x_3,x_4} \mathcal{L}^{\varepsilon,t}_{4,l}(x_1,\ldots,x_4)\Big[\int_0^{s_{12}} dr\;\;\dot{x}^{\nu}_{12}(r)\;\big(\nabla_\nu\varphi_2\big)(x_{12}(r))\;\varphi_3(x_3)\varphi_4(x_4) +$$

$$\varphi_2(x_1) \int_0^{s_{13}} dr \;\; \dot{x}_{13}^{\nu}(r) \; \big(\nabla_\nu \varphi_3\big)(x_{13}(r)) \; \varphi_4(x_4) \; +$$

$$\varphi_2(x_1) \; \varphi_3(x_1) \int_0^{s_{14}} dr \;\; \dot{x}_{14}^{\nu}(r) \; \big(\nabla_\nu \varphi_4\big)(x_{14}(r))\Big] \, . \tag{57}$$

Reparametrizing the geodesic segment $x_{12}(r) = X(\rho)$, $r = d(x_1, x_2)\rho$, $0 \leq \rho \leq 1$, we can rewrite the remainder (56) employing (A.30)

$$\ell_{2,l}^{\varepsilon,t}(x_1, \varphi) = \int_{x_2} d^{\,3}(x_1, x_2) \mathcal{L}_{2,l}^{\varepsilon,t}(x_1, x_2) \int_0^1 d\rho \frac{(1-\rho)^{\,2}}{2!\, d^3(x_1, x_2)} \dot{X}^{\nu_3}(\rho)\, \dot{X}^{\nu_2}(\rho)\, \dot{X}^{\nu_1}(\rho)\, \omega^{(3)}_{\nu_3\nu_2\nu_1}(X(\rho))$$

$$\text{where} \qquad \omega^{(3)}_{\nu_3\nu_2\nu_1}(x) \; = \; \big(\nabla_{\nu_3}\nabla_{\nu_2}\nabla_{\nu_1}\varphi\big)(x) \; . \tag{58}$$

1) **Boundary conditions** at $t = \varepsilon$:

The bare interaction (25) implies that at $t = \varepsilon$ - with $\mathcal{L}^{\varepsilon} \equiv \mathcal{L}^{\varepsilon,\varepsilon}$ -

$$\mathcal{L}_{n,l}^{\varepsilon}(x_1, \ldots x_n) \equiv 0 \;\; \text{for} \quad n > 4 \;, \quad \mathcal{L}_{2,0}^{\varepsilon} \equiv 0 \tag{59}$$

$$\mathcal{L}_{2,l}^{\varepsilon}(x_1, x_2) = \tilde{a}_l^{\epsilon}(x_1)\, \tilde{\delta}(x_2, x_1) - \; \Delta_2^{(b)}\, \tilde{\delta}(x_2, x_1) \, , \, b = b_l^{\mu\nu,\varepsilon}(x_2) \, , \tag{60}$$

$$\mathcal{L}_{4,l}^{\varepsilon}(x_1, \ldots x_4) = (\delta_{l,0} \;\, \lambda(x_1) \; + \; (1 - \delta_{l,0})\, c_l^{\varepsilon}(x_1)) \;\, \tilde{\delta}(x_2, x_1)\, \tilde{\delta}(x_3, x_1)\, \tilde{\delta}(x_4, x_1) \; . \tag{61}$$

To cope with the relevant part of the expansion (52) we consider a corresponding bare part

$$\mathcal{L}_{2,\,l}^{\varepsilon}(x_1, \varphi) \; = \; a_l^{\varepsilon}(x_1) \;\, \varphi(x_1) \; - \; f_l^{\mu,\,\varepsilon}(x_1) \;\, (\nabla_\mu \varphi)(x_1) \; - \; b_l^{\mu\nu,\,\varepsilon}(x_1)(\nabla_\mu \nabla_\nu \varphi)(x_1) \, . \tag{62}$$

The identity

$$- \; b_l^{\mu\nu}(x)(\nabla_\mu \nabla_\nu \varphi)(x) \; = \; \nabla_\nu \, b_l^{\mu\nu}(x) \cdot \big(\nabla_\mu \varphi\big)(x) \; - \; \Delta^{(b)}\varphi(x) \, , \quad \; b = b_l^{\mu\nu}(x) \tag{63}$$

suggests to decompose the bare vector coefficient appearing in (62) as

$$f_l^{\mu,\,\varepsilon}(x_1) \; = \; \tilde{f}_l^{\mu,\,\varepsilon}(x_1) \; + \; \nabla_\nu \, b_l^{\mu\nu,\,\varepsilon}(x_1) \, . \tag{64}$$

By folding (62) with a test function φ we obtain after partial integration [13]

$$\int_x \varphi(x) \, \mathcal{L}_{2,\,l}^{\varepsilon}(x, \varphi) \; = \; \int_x \Big\{ \big(a_l^{\varepsilon}(x) + \frac{1}{2} \, \nabla_\mu \tilde{f}_l^{\mu,\,\varepsilon}(x) \big) \, \varphi^2(x) + \, b_l^{\mu\nu,\varepsilon}(x) \;\, \partial_\mu \varphi(x) \cdot \partial_\nu \varphi(x) \Big\} \, . \tag{65}$$

This agrees in form with the corresponding content of the bare action (25). From the boundary conditions (59)-(61) we deduce

$$\ell_{2,l}^{\varepsilon,\varepsilon} \;\, (x_1, \, \varphi) \;\; = \;\; 0 \;\, , \quad \ell_{4,l}^{\varepsilon,\varepsilon} \;\, (x_1, \, \varphi) \;\; = \;\; 0 \; . \tag{66}$$

[13] Here φ is assumed to be smooth, and to decrease sufficiently rapidly if $\mathcal{M}$ is noncompact. Apart from the present consideration and from the general analysis of the effective action after (20), we do not introduce test functions against which the first (root) vertex is integrated.

The renormalization problem is related to the behaviour of the heat kernel at small values of t. Therefore this problem is essentially solved if we can integrate the flow equations up to some finite value t_R of t. For shortness we choose units such that $t_R = 1$. We will come to the limit $t \to \infty$ later, see Proposition 3. The positive mass $m > 0$ only plays a role when this limit is taken. We pose

2) **Renormalization conditions** at $t = t_R := 1$: [14]

$$a_l^{\varepsilon,1}(x_1) := a_l^R(x_1), \quad f_l^{\mu,\varepsilon,1}(x_1) := f_l^{\mu,R}(x_1), \quad b_l^{\mu\nu,\varepsilon,1}(x_1) := b_l^{\mu\nu,R}(x_1)\,, \tag{67}$$

$$c_l^{\varepsilon,1}(x_1) := c_l^R(x_1)\,, \tag{68}$$

where $b_l^{\mu\nu,R}(x)$ is a smooth symmetric tensor of type $(2,0)$, $f_l^{\mu,R}(x)$ is a smooth vector and $a_l^R(x)$, $c_l^R(x)$ are smooth scalars on $\mathcal{M}$, all uniformly bounded in the norm (A.32). Typically the renormalization conditions are assumed to be cutoff-independent. To be able to analyse the relation between the bare (inter)action and the renormalization conditions in more detail later on, we shall be more general in also admitting weakly ε-dependent renormalization functions satisfying

$$|\partial_\varepsilon\, a_l^R(x)| \,<\, O(\varepsilon^{-\eta})\ , \quad |a_l^R(x)| \,<\, const \,+\, O(\varepsilon^{1-\eta})\ , \quad \eta \le 1/2\ , \tag{69}$$

with analogous expressions for the other renormalization functions.

In the particular case of $\mathcal{M}$ having constant curvature, i.e. where all sectional curvatures of $\mathcal{M}$ have a constant value ρ, a transitive isometry group G acts on $\mathcal{M}$. There are three types of such manifolds: The sphere $\mathcal{S}^4$ with $\rho = k^2$ and $G = SO(5)$, the flat space $\mathbf{R}^4$ with $\rho = 0$ and $G = SO(4) \otimes_s \mathbf{R}^4$, the hyperbolic space $\mathcal{H}^4$ with $\rho = -k^2$ and $G = SO_0(4,1)$, the subscript denoting the component connected to the identity. Requiring the Schwinger functions to show this symmetry G, results in the following restrictions on the relevant terms:

i) $a_l^{\varepsilon,t}(x)$, $c_l^{\varepsilon,t}(x)$ do not depend on $x \in \mathcal{M}$,

ii) $f_l^{\mu,\varepsilon,t}(x) \equiv 0\,, \quad b_l^{\mu\nu,\varepsilon,t}(x) = g^{\mu\nu}(x)\, b_l^{\varepsilon,t}$, hence $\nabla_\nu\, b_l^{\mu\nu,\varepsilon,t}(x) \equiv 0$.

However, there is a further (dimensionless) parameter $\zeta = k^2/m^2$ on which $a_l^{\varepsilon,t}$, $b_l^{\varepsilon,t}$, $c_l^{\varepsilon,t}$ may depend, in general.

6 Renormalizability

The subsequent proposition is proven for test functions of the form $\varphi_{\tau_{2,s},y_{2,s}}(x_{2,n})$, (42). In the end of this section we join some remarks on possible extensions of the class of

[14] The scale t_R is related to the scale T appearing in the bounds on the heat kernel (13), (14), (15) .

test functions. By Bose symmetry the bounds stay unaltered if any permuted subset of external coordinates (and not $x_2, \ldots, x_s$) is folded with test functions.

Proposition 1:

We consider $0 < \varepsilon \leq t \leq 1$ *and* $\varepsilon < \tau_i$ *, furthermore* $1 \leq s \leq n$ *,* $2 \leq i \leq n$ *,* $2 \leq j \leq s$ *and* $0 \leq r \leq 3$ *. We consider test functions either of the form* $\varphi_{\tau_{2,s}, y_{2,s}}(x_{2,n})$ *or* $\varphi^{(j)}_{\tau_{2,s}, y_{2,s}}(x_{2,n})$ *, which are also denoted in shorthand as* φ_s *resp.* $\varphi^{(j)}_s$ *.*

In all subsequent bounds we understand $\mathcal{P}_l$ *to denote a polynomial of degree* $\leq \sup(l, 0)$ *- each time it appears possibly a new one - with nonnegative coefficients which may depend on* l, n, δ [15]*, on* $\sup_{\mathcal{M}} |\lambda(x)|$ *, as well as on* k^2 *,* κ^2 *and the bounds on the first and second covariant derivatives of the curvature tensor (see (13) - (15)), but not on* ε*,* t*,* m *and* τ*. Constants* $O(1)$ *in the subsequent proof are to be understood in the same way. By* (t,τ) *we denote* $\inf\{\tau_2, \ldots, \tau_s, t\}$*, by* $(t,\tau)_i$ *we denote* $\inf\{\tau_2, \ldots, \cancel{\tau_i}, \ldots, \tau_s, t\}$*.*

Then we claim the following bounds - using the shorthand (47) -

$$|\,\mathcal{L}^{\varepsilon,t}_{n,l}(x_1, \varphi_{\tau, y_{2,s}})| \leq t^{\frac{n-4}{2}} \, \mathcal{P}_l \log(t,\tau)^{-1} \, \mathcal{F}_{s,l}(t,\tau) \ , \quad n \geq 4 \tag{70}$$

$$|\,\mathcal{L}^{\varepsilon,t}_{n,l}(x_1, E^{(r)}_{(i)} \, \varphi_{\tau, y_{2,s}})| \leq |\,\omega^{(r)}(x_1)| \, t^{\frac{n+r-4}{2}} \, \mathcal{P}_l \log(t,\tau)^{-1}_i \, \mathcal{F}_{s,l}(t,\tau) \ , \quad n > 4, \ r > 0 \tag{71}$$

$$|\,\mathcal{L}^{\varepsilon,t}_{n,l}(x_1, E^{(r)}_{(i)} \varphi_{\tau, y_{2,s}})| \leq |\,\omega^{(r)}(x_1)| \, t^{\frac{n+r-4}{2}} \, \mathcal{P}_{l-1} \log(t,\tau)^{-1}_i \, \mathcal{F}_{s,l}(t,\tau) \tag{72}$$

$$n = 2, r = 3 \ \ or \ \ n = 4, \ r > 0$$

$$|\,\mathcal{L}^{\varepsilon,t}_{2,l}(x_1, \varphi_{\tau, y_2})| \leq (t,\tau)^{-1} \, \mathcal{P}_{l-1} \log(t,\tau)^{-1} \, \mathcal{F}_{2,l}(t,\tau) \tag{73}$$

$$|\,\mathcal{L}^{\varepsilon,t}_{2,l}(x_1, E_{(2)} \, \varphi_{\tau, y_2})| \leq |\,\omega(x_1)| \, (t,\tau)^{-1/2} \, \mathcal{P}_{l-1} \log(t,\tau)^{-1} \, \mathcal{F}_{2,l}(t,\tau) \tag{74}$$

$$|\,\mathcal{L}^{\varepsilon,t}_{2,l}(x_1, E^{(2)}_{(2)} \, \varphi_{\tau, y_2})| \leq |\,\omega^{(2)}(x_1)| \, \mathcal{P}_{l-1} \log(t,\tau)^{-1} \, \mathcal{F}_{2,l}(t,\tau) \tag{75}$$

$$|\,\mathcal{L}^{\varepsilon,t}_{n,l}(x_1, \varphi^{(j)}_{\tau, y_{2,s}})| \leq t^{\frac{n-4}{2}} \left(\frac{t}{\tau_j}\right)^{1/2} \mathcal{P}_l \log(t,\tau)^{-1} \, \mathcal{F}_{s,l}(t,\tau) \ , \quad n > 2 \tag{76}$$

$$|\,\mathcal{L}^{\varepsilon,t}_{2,l}(x_1, \varphi^{(2)}_{\tau, y_2})| \leq (\frac{t}{\tau})^{1/2} (t,\tau)^{-1} \, \mathcal{P}_{l-1} \log(t,\tau)^{-1} \, \mathcal{F}_{2,l}(t,\tau) \tag{77}$$

$$|\,F^{(r)}_{(12)} \mathcal{L}^{\varepsilon,t}_{n,l}(x_1, x_2, \varphi)| \leq |\,\omega^{(r)}(x_1)| \, t^{\frac{n-1}{2}} \, \mathcal{P}_{l-1} \log(t,\tau)^{-1} \, \mathcal{F}^{(12)}_{s,l}(t,\tau) \ . \tag{78}$$

$$r = 0, 1, 2 \quad and \ |\,\omega^{(0)}(x_1)| \equiv 1 \, .$$

Remark : The full series of the previous bounds is needed to close the inductive argument in the subsequent proof. The reader who only wants to know what the bounds are can restrict to (70), (73).

[15] We suppose that $\delta > 0$ may be chosen arbitrarily small in the definition of $\mathcal{F}$. The constants in $\mathcal{P}_l$ then depend on the choice of δ.

Proof : The bounds stated in the proposition are proven inductively using the (standard) inductive scheme which proceeds upwards in $n+2l$, and for given $n+2l$ upwards in l. Thus the induction starts with the pair $(4,0)$. For this term the r.h.s. of the FE vanishes so that $\mathcal{L}^{\varepsilon,t}_{4,0}(x_1,\ldots,x_4) = \lambda(x_1)\,\tilde\delta(x_2,x_1)\,\tilde\delta(x_3,x_1)\,\tilde\delta(x_4,x_1)$ which is compatible with our bounds (after folding with suitable φ). Generally it is important to note that the boundary conditions are compatible with the bounds of the proposition.
We will first derive bounds for the derivatives $\partial_t\,\mathcal{L}^{\varepsilon,t}_{n,l}(x_1,E^{(r)}_{(i)}\varphi_s)$, where $E^{(0)}_{(i)}\equiv 1$, and $\partial_t\,\mathcal{L}^{\varepsilon,t}_{n,l}(x_1,\varphi_s^{(j)})$. Afterwards these bounds are integrated over w.r.t. t.
A) We start considering the case $r=0$ and test functions φ_s.
A1) Here we first treat the first term on the r.h.s. of (32)

$$R_1 := \int_{x_{2,n},x,y} \varphi_s(x_{2,n})\ C_t(x,y)\ \mathcal{L}^{\varepsilon,t}_{n+2,l-1}(\vec{x}_n,x,y)\ .$$

We may rewrite this expression as

$$R_1 = \int_v\int_{x_2,\ldots,x_n,x,y} \varphi_s(x_{2,n})\ C_{t/2}(x,v)\ C_{t/2}(v,y)\ \mathcal{L}^{\varepsilon,t}_{n+2,l-1}(\vec{x}_n,x,y)\ =$$

$$\int_v \mathcal{L}^{\varepsilon,t}_{n+2,l-1}(x_1,\varphi_s\times C_{t/2}(\cdot,v)\times C_{t/2}(v,\cdot))\ .$$

Applying the induction hypothesis to $\mathcal{L}^{\varepsilon,t}_{n+2,l-1}(x_1,\varphi_s\times C_{t/2}(\cdot,v)\times C_{t/2}(v,\cdot))$ we thus obtain the bound

$$|R_1| \le t^{\frac{n+2-4}{2}}\ \mathcal{P}_{l-1}\log(t,\tau)^{-1}\int_v\int_{\vec z}\sum_{T^{s+2}_{l-1}(x_1,y_{2,s},v,v)} \mathcal{F}(t,\tau,\frac{t}{2},\frac{t}{2};T^{s+2}_{l-1}(x_1,y_{2,s},v,v,\vec z))\ . \quad (79)$$

For any contribution to (79) we denote by z', z'' the vertices in the respective tree $T^{s+2}_{l-1}(x_1,y_{2,s},v,v,\vec z)$ to which the test functions $C_{t/2}(\cdot,v)$, $C_{t/2}(v,\cdot)$ are attached. Interchanging $\int_{\vec z}$ (see (46)) and $\int_v$ and performing the integral over v using (5), (19), we get a contribution

$$\int_v C_{t/2}(z',v)\ C_{t/2}(v,z'')\ =\ C_t(z',z'')\ \le\ O(1)\ t^{-2}\ . \quad (80)$$

Using this bound we can majorize $\int_v \mathcal{F}(t,\tau,\frac{t}{2},\frac{t}{2};T^{s+2}_{l-1}\ ;x_1,y_{2,s},v,v)$ by

$$O(1)\ t^{-2}\ \mathcal{F}(t,\tau;T^s_l\ ;x_1,y_{2,s})$$

where the tree T^s_l is the *reduced tree* obtained from T^{s+2}_{l-1} by taking away the two external lines ending in v, see Definition 4. The reduction process for each tree fixes uniquely the set of internal vertices of T^s_l in terms of those of T^{s+2}_{l-1}. Note that the elimination of

vertices of incidence number 1 together with their adjacent line is justified by the fact that $\int_{z'} C_{t_I,\delta_l}(z,z') \le 1$. Note also that in the tree T^s_l the number v_2 of vertices of incidence number 2 may have increased by at most 2, as compared to T^{s+2}_{l-1}, so that T^s_l is indeed an element from $\mathcal{T}^s_l$. We keep track of this lower index l in the tree basically to show that the number of vertices always stays finite (in fact does not grow faster than linearly in l for n fixed).
The final bound for the first term on the r.h.s. of the FE is thus

$$|R_1| \le t^{\frac{n-6}{2}} \ \mathcal{P}_{l-1}\log(t,\tau)^{-1} \sum_{T^s_l(x_1,y_{2,s})} \mathcal{F}(t,\tau;T^s_l\ ;x_1,y_{2,s}) \tag{81}$$

where constants have been absorbed in $\mathcal{P}_{l-1}\log$.

A2) We now consider the second term in (32) for
i) $n>4$
Picking a generic term from the symmetrized sum and arguing as in A1) we have to bound

$$R_2 := \int_{x_{2,n},x,y} \varphi_s(x_{2,n})\ C_t(x,y)\ \mathcal{L}^{\varepsilon,t}_{n_1+1,l_1}(x_1,\dots,x_{n_1},x)\ \mathcal{L}^{\varepsilon,t}_{n_2+1,l_2}(y,x_{n_1+1},\dots,x_n)$$

which we rewrite similarly as in A1)

$$R_2 = \int_v \int_{x_2,\dots,x_n,x,y} \varphi_s(x_{2,n})\ C_{t/2}(x,v)\ C_{t/2}(v,y)\ \mathcal{L}^{\varepsilon,t}_{n_1+1,l_1}(x_1,\dots,x)\ \mathcal{L}^{\varepsilon,t}_{n_2+1,l_2}(y,\dots,x_n)\ . \tag{82}$$

Denoting

$$\varphi'_{s_1}(x_{2,n_1}) = \prod_{r=2}^{n_1} \varphi_r(x_r)\ , \quad \varphi''_{s_2}(x_{n_1+1,n-1}) = \prod_{r=n_1+1}^{n-1} \varphi_r(x_r)$$

we identify the two terms

$$\int_{x_2,\dots,x_{n_1},x} \varphi'_{s_1}(x_{2,n_1})\ C_{t/2}(x,v)\ \mathcal{L}^{\varepsilon,t}_{n_1+1,l_1}(x_1,\dots,x)$$

and

$$\int_{x_{n_1+1},\dots,x_n,y} \varphi''_{s_2}(x_{n_1+1,n-1})\ C_{t/2}(v,y)\ \mathcal{L}^{\varepsilon,t}_{n_2+1,l_2}(y,\dots,x_n)$$

and thus write (82) as

$$R_2 = \int_{x_n}\int_v \mathcal{L}^{\varepsilon,t}_{n_1+1,l_1}(x_1,\varphi'_{s_1}\times C_{t/2}(\cdot,v))\ \mathcal{L}^{\varepsilon,t}_{n_2+1,l_2}(x_n,\ C_{t/2}(v,\cdot)\times\varphi''_{s_2})\ \ \varphi_n(x_n)\ . \tag{83}$$

On applying the induction hypothesis to both terms in (83), restricting first to $n_1, n_2 > 1$, we obtain the bound

$$|R_2| \le t^{\frac{n+2-8}{2}} \mathcal{P}_{l_1} \log(t,\tau)^{-1} \int_{x_n} \int_v \sum_{T^{s_1+1}_{l_1},\ T^{s_2+1}_{l_2}} \mathcal{F}(t,\tau',t/2; T^{s_1+1}_{l_1}\ ; x_1, y_2, \ldots, y_{s_1}, v) \cdot$$

$$\cdot\ \mathcal{P}_{l_2} \log(t,\tau)^{-1}\ \mathcal{F}(t,\tau'',t/2; T^{s_2+1}_{l_2}\ ; x_n, v, y_{s_1+1}, \ldots\ldots, y_{s(n)})\ \ \varphi_n(x_n)\ . \tag{84}$$

Here $s(n) = s$ if $s < n$, and $s(n) = s-1$ if $s = n$. Interchanging the integral over v with the sum over trees we obtain

$$|R_2| \le t^{\frac{n-6}{2}}\ \mathcal{P}_l \log(t,\tau)^{-1} \sum_{T^s_l(T^{s_1+1}_{l_1},\ T^{s_2+1}_{l_2})(x_1,y_2\ldots,y_s)} \int_{x_n} \mathcal{F}(t,\tau_{2,s}; T^s_l\ ; x_1, y_2 \ldots, y_s) \tag{85}$$

with the following explanations :
Any contribution in the sum over trees $T^s_l(T^{s_1+1}_{l_1},\ T^{s_2+1}_{l_2})(x_1, y_2, \ldots, y_s, \vec{z})$ is obtained from $T^{s_1+1}_{l_1}(x_1, y_2 \ldots, y_{s_1}, v, \vec{z}\,')$ and $T^{s_2+1}_{l_2}(x_n, v, y_{s_1+1}, \ldots, y_s, \vec{z}\,'')$ by joining these two trees via the lines going from the vertices z' and z'' to v, where z' and z'' are the vertices attached to v in the two trees. These two lines have parameters $t/2$. We use the equality

$$\int_v C_{t/2}(z',v)\ C_{t/2}(v,z'')\ =\ C_t(z',z'') \tag{86}$$

so that the new internal line has a t-parameter in the interval $[\varepsilon, t]$ over which the sup is taken in the definition of $\mathcal{F}$. [16]
When performing the integral over x_n in (84) we remember that x_n is the root vertex of $T^{s_2+1}_{l_2}(x_n, v, y_{s_1+1}, \ldots\ldots, y_s, \vec{z})$. If $s = n$ we have $\varphi_n(x_n) = C_{\tau_n}(x_n, y_n)$, and x_n becomes an internal vertex, and y_n an external vertex, of T^s_l. If $s < n$, then $\varphi_n(x_n) \equiv 1$, and the vertex x_n becomes an internal vertex of T^s_l unless $c(x_n) = 1$. In this last case integration over x_n together with (4) permits to take away the vertex x_n and the internal line joining it to an internal vertex z_j of the tree $T^{s_2+1}_{l_2}$ [17]. If (originally) $c(z_j) = 2$ this elimination process continues. Thus the final bound for R_2, and hence for the second term in (32) is the same as (81), apart from changing $\mathcal{P}_{l-1} \log(t,\tau)^{-1} \to \mathcal{P}_l \log(t,\tau)^{-1}$. Note that this bound is established in the same way if $\mathcal{L}^{\varepsilon,t}_{n_1+1,l_1}(x_1, \ldots, x)$ or $\mathcal{L}^{\varepsilon,t}_{n_2+1,l_2}(y, \ldots, x_n)$ are two-point functions : In this case the parameter τ appearing in (73) equals $t/2$ so that $(t,\tau)^{-1}$ can be replaced by $2/t$.

[16] We can of course majorize $C_t(z',z'') \le O(1)\ C_{t_\delta}(z',z'')$.
[17] If x_n has turned into a vertex of incidence number 2 for T^s_l, the bound $v_2 + \delta_{c_1,1} \le 3l - 2 + s/2$ is easily verified.

Taking both contributions from the r.h.s. of the FE together and summing over all trees we have established the bounds

$$|\partial_t \mathcal{L}^{\varepsilon,t}_{n,l}(x_1,\varphi_s)| \ \le\ t^{\frac{n-6}{2}}\ \mathcal{P}_l \log(t,\tau)^{-1}\ \mathcal{F}_{s,l}(t,\tau)\ , \quad n>4\ . \tag{87}$$

ii) $n \le 4$
In this case we have $n_1+1=2$ and/or $n_2+1=2$. Thus at least one of the polynomials $\mathcal{P}_{l_i}\log(t,\tau)^{-1}$ appearing in the bounds (84) can by induction be replaced by $\mathcal{P}_{l_i-1}\log t^{-1}$. Therefore proceeding exactly as in the previous case we obtain the bounds

$$|\partial_t \mathcal{L}^{\varepsilon,t}_{n,l}(x_1,\varphi_s)| \ \le\ t^{\frac{n-6}{2}}\ \mathcal{P}_{l-1} \log(t,\tau)^{-1}\ \mathcal{F}_{s,l}(t,\tau)\ , \quad n\le 4\ . \tag{88}$$

B) $r>0$, cf.(40)
For the first term on the r.h.s. of the flow equation resulting from (32) the bounds for $1\le r\le 3$ are proven exactly as in A1). For the second term we proceed similarly as in A2). We pick a generic term on the r.h.s.

$$\int_{x_{2,n},x,y} \varphi_s(x_{2,n})\ C_t(x,y)\ E(x_k,x_1\ ;\omega^{(r)})\mathcal{L}^{\varepsilon,t}_{n_1+1,l_1}(x_1,\ldots,x_{n_1},x)\ \mathcal{L}^{\varepsilon,t}_{n_2+1,l_2}(y,x_{n_1+1},\ldots,x_n)\ .$$

In the case where $k\le n_1$ the proof is the same as for $r=0$, up to inserting the modified induction hypothesis for

$$\int_{x_{2,n_1},x} \varphi_{s_1}(x_{2,n_1})\ C_{t/2}(x,v)\ E(x_k,x_1\ ;\omega^{(r)})\mathcal{L}^{\varepsilon,t}_{n_1+1,l_1}(x_1,\ldots,x_{n_1},x)\ .$$

If $k>n_1$ we assume without restriction $k=n$ and proceed again as in A2) to obtain the bound

$$t^{\frac{n-6}{2}}\ \mathcal{P}_{l_1}\log(t,\tau)^{-1}_n\ \int_{x_n}\int_v |E(x_n,x_1\ ;\omega^{(r)})| \sum_{T^{s_1+1}_{l_1},\ T^{s_2+1}_{l_2}} \mathcal{F}(t,\tau,t/2;T^{s_1+1}_{l_1}\ ;x_1,y_2,\ldots,y_{s_1},v)$$

$$\cdot\ \mathcal{P}_{l_2}\log(t,\tau)^{-1}_n\ \mathcal{F}(t,\tau,t/2;T^{s_2+1}_{l_2}\ ;x_n,v,y_{s_1+1},\ldots\ldots,y_{s(n)})\ \varphi_n(x_n)\ . \tag{89}$$

Observing the inequality (39) together with

$$d(x_n,x_1)\le \sum_{a=1}^{q} d(v_a,v_{a-1}) \tag{90}$$

where $\{v_a\}$ are the positions of the internal vertices in the tree $T^s_l(T^{s_1+1}_{l_1},\,T^{s_2+1}_{l_2})$ defined as in A2), on the path joining $x_1=v_0$ and $x_n=v_q$, we then use the bound (14). The

cases $s=n$ and $s<n$ are treated as in A2), using once more the bound (14).
The previous reasoning holds a fortiori for $\partial_t\, F^{(r)}_{(12)}\mathcal{L}^{\varepsilon,t}_{n,l}(x_1,x_2,\varphi)$, since in these cases we have $|\,F^{(r)}_{(12)}\,|\le d^3(x_1,x_2)|\,\omega^{(r)}(x_1)|\,,\,|\,\omega^{(0)}(x_1)\,|\equiv 1$. Here then x_2 takes the role of x_n.
Proceeding as before we thus obtain for $r\neq 0$ (after absorbing again all constants in $\mathcal{P}_l$)

$$|\partial_t\,\mathcal{L}^{\varepsilon,t}_{n,l}(x_1,E^{(r)}_{(i)}\,\varphi_s)|\;\le\;|\omega^{(r)}(x_1)|\,t^{\frac{n+r-6}{2}}\;\mathcal{P}_l\log(t,\tau)^{-1}_i\;\mathcal{F}_{s,l}(t,\tau)\;,\quad n>4 \tag{91}$$

$$|\partial_t\,\mathcal{L}^{\varepsilon,t}_{4,l}(x_1,E^{(r)}_{(i)}\,\varphi_s)|\;\le\;|\omega^{(r)}(x_1)|\,t^{\frac{r-2}{2}}\;\mathcal{P}_{l-1}\log(t,\tau)^{-1}_i\;\mathcal{F}_{s,l}(t,\tau) \tag{92}$$

$$|\partial_t\,\mathcal{L}^{\varepsilon,t}_{2,l}(x_1,E^{(r)}_{(2)}\,\varphi_2)|\;\le\;|\,\omega^{(r)}(x_1)|\;t^{\frac{r-4}{2}}\;\mathcal{P}_{l-2}\log t^{-1}\;\mathcal{F}_{2,l}(t,\tau) \tag{93}$$

$$|\,\partial_t\,F^{(r)}_{(12)}\mathcal{L}^{\varepsilon,t}_{n,l}(x_1,x_2,\varphi)|\;\le\;|\,\omega^{(r)}(x_1)|\;t^{\frac{n-3}{2}}\;\mathcal{P}_{l-1}\log(t,\tau)^{-1}\;\mathcal{F}^{(12)}_{s,l}(t,\tau)\;. \tag{94}$$

In (94) τ stands for $(\tau_3,\ldots,\tau_s)$, furthermore $r=0,1,2$ and $|\,\omega^{(0)}(x_1)|\equiv 1$.

The bounds for (54)-(55),

$$|\,\partial_t\,c^{\varepsilon,t}_l(x_1)\,|\le\;\frac{1}{t}\;\mathcal{P}_{l-1}\log\frac{1}{t}\;,\quad |\,\partial_t\,a^{\varepsilon,t}_l(x_1)\,|\le\;t^{-2}\;\mathcal{P}_{l-1}\log\frac{1}{t}\;, \tag{95}$$

$$|\,\partial_t\,f^{\mu,\varepsilon,t}_l(x_1)\,\omega_\mu(x_1)|\;\;\le\;\;|\,\omega(x_1)|\;\;t^{-3/2}\;\mathcal{P}_{l-2}\log\frac{1}{t}\;, \tag{96}$$

$$|\,\partial_t\;b^{\mu\nu,\varepsilon,t}_l(x_1)\,\omega^{(2)}_{\mu\nu}(x_1)\,|\;\;\le\;\;|\,\omega^{(2)}(x_1)|\,\frac{1}{t}\;\mathcal{P}_{l-2}\log\frac{1}{t} \tag{97}$$

are obtained on restricting the previous considerations to the case $s=1$, in which all external coordinates are integrated over, e.g.

$$\partial_t\,a^{\varepsilon,t}_l(x_1)\;=\;\frac{1}{2}\int_{x_2,x,y}C_t(x,y)\left\{\mathcal{L}^{\varepsilon,t}_{4,l-1}(x_1,x_2,x,y)\;-\sum_{l_1+l_2=l}\Big[\mathcal{L}^{\varepsilon,t}_{2,l_1}(x_1,x)\,\mathcal{L}^{\varepsilon,t}_{2,l_2}(y,x_2)\Big]_{sym}\right\}.$$

The polynomials appearing in (96), (97) are of degree $\le\,l-2$, corresponding to the fact that on the r.h.s. of the FE (32) for these terms, there appear $\mathcal{L}^{\varepsilon,t}_{l-1,4}$ and $\mathcal{L}^{\varepsilon,t}_{l_1,2}\,\mathcal{L}^{\varepsilon,t}_{l_2,2}$ with insertions $E^{(r)}_{(2)}$, $r=1,2$. Both are bounded inductively by polynomials of total degree $\le\sup(l-2,0)$.

C) We come to the bound on $\partial_t\mathcal{L}^{\varepsilon,t}_{n,l}(x_1,\varphi^{(j)}_s)$, cf. (76). As compared to B) the only case which requires new analysis is the bound on the second term from the r.h.s. of the FE (32), in the case $j>s_1$. Then we assume without restriction, similarly as in B), that $j=s$. The term to be bounded corresponding to (84) is then

$$t^{\frac{n-6}{2}}\;\mathcal{P}_{l_1}\log(t,\tau)^{-1}\;\int_v\sum_{T^{s_1+1}_{l_1},\;T^{s_2+1}_{l_2}}\mathcal{F}(t,\tau',t/2;T^{s_1+1}_{l_1}\;;x_1,y_2,\ldots,y_{s_1},v)\;\cdot$$

$$\cdot\ \mathcal{P}_{l_2} \log(t,\tau)^{-1} \int_{x_s} \mathcal{F}(t,\tau'',t/2;T_{l_2}^{s_2+1}\ ;x_s,v,y_{s_1+1},\ldots\ldots,y_{s-1})\ |K^{(1)}(\tau_s,x_s,x_1;y_s)|\ . \tag{98}$$

To bound this expression we telescope the difference $K^{(1)}(\tau_s,x_s,x_1;y_s)$, cf.(44), along the tree $T_l^s(T_{l_1}^{s_1+1},T_{l_2}^{s_2+1})$ obtained from the two initial trees by joining them via v as in A2) and proceeding similarly as in (90). We then have to bound expressions of the type

$$C_{t_I,\delta}(v_{a-1},v_a)\,|K(\tau_s,v_a,y_s)\ -\ K(\tau_s,v_{a-1},y_s)|$$

where v_{a-1},v_a are adjacent internal vertices in $T_l^s(T_{l_1}^{s_1+1},T_{l_2}^{s_2+1})$ on the unique path from x_1 to y_s. Making use of the covariant Schlömilch formula (A.28)-(A.31) for the difference $K^{(1)}(\tau_s,v_a,v_{a-1};y_s)$, we obtain

$$\begin{aligned}
|K(\tau_s,v_a,y_s)\ -\ K(\tau_s,v_{a-1},y_s)| &\leq \int_0^s dr\,|\,\nabla_{(1)}K(\tau_s,z(r),y_s)\,| \\
&= d(v_{a-1},v_a)\int_0^1 d\rho\,|\,\nabla_{(1)}K(\tau_s,v_{a-1,a}(\rho),y_s)\,| \\
&\leq O(1)\,\frac{d(v_{a-1},v_a)}{\sqrt{\tau_s}}\int_0^1 d\rho\,K(\tau_{s,\delta'},v_{a-1,a}(\rho),y_s)
\end{aligned} \tag{99}$$

where $z(r)=v_{a-1,a}(\rho)$ lies at distance $r=\rho\,d(v_{a-1},v_a)$, $0\leq\rho\leq 1$, from v_{a-1} on the reparametrized geodesic segment from v_{a-1} to v_a. The last inequality results from (15). Introducing for

$$3\,\delta<1: \qquad b=2\,\frac{1+3\delta}{1-3\delta}\ , \tag{100}$$

we then bound, with $\delta'>0$ to be fixed later,

$$\begin{aligned}
&C_{t_I,\delta}(v_{a-1},v_a)\,|\,K(\tau_s,v_a,y_s)\ -\ K(\tau_s,v_{a-1},y_s)|\ \leq \\
&\leq \begin{cases} C_{t_I,\delta}(v_{a-1},v_a)\,K(\tau_s,v_a,y_s)\ +C_{t_I,\delta}(v_{a-1},v_a)\,K(\tau_s,v_{a-1},y_s)\ , & b\,t\geq\delta'\tau_s \\ O(1)\ C_{t_I,2\delta}(v_{a-1},v_a)\,(\frac{t_I}{\tau_s})^{1/2}\ \int_0^1 d\rho\ K(\tau_{s,\delta'},v_{a-1,a}(\rho),y_s)\ , & b\,t<\delta'\tau_s\ . \end{cases}
\end{aligned} \tag{101}$$

The last line follows using (99) and absorbing the factor $d(v_{a-1},v_a)$ in $C_{t_I,\delta}(v_{a-1},v_a)$ with the aid of (14), by changing δ to 2δ .

The last line in (101) has to be bounded in such a way as to reproduce a contribution compatible with the induction hypothesis. To this end we use the (upper) bound (13)

$$C_{t_{2\delta}}(v_1,v_2)\ K(\tau_{\delta'},v_{1,2}(\rho),y)\ \leq\ O(1)\,\frac{1}{t^2}\,\frac{1}{\tau^2}\ \exp\left(-\frac{d^2(v_1,v_2)}{4t(1+3\delta)}\ -\frac{d^2(v_{1,2}(\rho),y)}{4\tau(1+\delta')^2}\right)\ .$$

Noting that $d(v_1,v_2)=d(v_1,v_{1,2}(\rho))+d(v_{1,2}(\rho),v_2)$ we deduce

$$\frac{1}{\delta'}\ d^2(v_1,v_2)+d^2(v_{1,2}(\rho),y)\geq\frac{1}{\delta'}\ d^2(v_1,v_{1,2}(\rho))+d^2(v_{1,2}(\rho),y)\geq$$

$$\frac{1}{1+\delta'}\ \left(d(v_1,v_{1,2}(\rho))+d(v_{1,2}(\rho),y)\right)^2 \geq \frac{1}{1+\delta'}\ d^2(v_1,y)\ .$$

Hence, observing (100), we find for $b\,t<\delta'\tau$

$$\begin{aligned}\frac{d^2(v_1,v_2)}{4t(1+3\delta)}+\frac{d^2(v_{1,2}(\rho),y)}{4\tau(1+\delta')^2} &= \frac{d^2(v_1,v_2)}{8t}+\frac{d^2(v_1,v_2)}{4bt}+\frac{d^2(v_{1,2}(\rho),y)}{4\tau(1+\delta')^2}\\ &\geq \frac{d^2(v_1,v_2)}{8t}+\frac{d^2(v_1,y)}{4\tau(1+\delta')^3}\ .\end{aligned}$$

With the aid of the lower bound (13) we then arrive at

$$C_{t_{I,2\delta}}(v_1,v_2)\int_0^1 d\rho\ K(\tau_{s,\delta'},v_{1,2}(\rho),y_s)\leq O(1)\,C_{2t_I,\delta}(v_1,v_2)\,K(\tau_s(1+\delta')^4,v_1,y)\ . \tag{102}$$

Taking into account (5) and choosing δ' such that $(1+\delta')^4=1+\delta$, i.e. $\delta'=\delta/4+O(\delta^2)$, we may thus bound the last line in (101) by

$$O(1)\ (\frac{t}{\tau_s})^{1/2}\ K(\tau_{s,\delta},v_{a-1},y_s)\ \int_v C_{t_I,\delta}(v_{a-1},v)\,C_{t_I,\delta}(v,v_a)\ ,\quad b\,t<\delta'\ \tau\ . \tag{103}$$

Note that the addition of a new internal vertex v of incidence number 2 in (103) is compatible with the inequality $v_2+\delta_{c_1,1}\leq 3l-2+s/2$.

Using these bounds and going back to (98) we realize that the two terms in the first line of (101) - case $b\,t\geq\delta'\tau_s$ - correspond to two new trees of type $\mathcal{T}_l^s$, where in comparison to $T_l^s(T_{l_1}^{s_1+1},T_{l_2}^{s_2+1})$ the incidence number of the vertex v_{a-1} or v_a has increased by one unit. Similarly (103) - case $b\,t<\delta'\tau_s$ - corresponds to a new tree where the incidence number of the vertex v_{a-1} has increased by one unit. In (98) an integral over x_s is performed. If in the new tree

i) x_s has $c(x_s)>1$ [18], then x_s takes the role of an internal vertex of the new tree,

ii) x_s has $c(x_s)=1$ we integrate over x_s using (4) so that the vertex x_s disappears.

As a consequence of the previous bounds, on replacing again $s\to j$ in (101),(103), we thus obtain for $n>4$ - see also A2) ii) for (105), (106) -

$$|\partial_t\,\mathcal{L}^{\varepsilon,t}_{n,l}(x_1,\varphi^{(j)}_{\tau_{2,s},y_{2,s}})|\ \leq\ \left\{\begin{array}{rl} t^{\frac{n-6}{2}}\ \mathcal{P}_l\log(t,\tau)^{-1}\ \mathcal{F}_{s,l}(t,\tau)\ , & b\,t\geq\delta'\tau_j\\ t^{\frac{n-6}{2}}\ (\frac{t}{\tau_j})^{1/2}\ \mathcal{P}_l\log(t,\tau)^{-1}\ \mathcal{F}_{s,l}(t,\tau)\ , & b\,t<\delta'\tau_j\end{array}\right. \tag{104}$$

$$|\partial_t\,\mathcal{L}^{\varepsilon,t}_{4,l}(x_1,\varphi^{(j)}_{\tau_{2,s},y_{2,s}})|\ \leq\ \left\{\begin{array}{rl} t^{-1}\ \mathcal{P}_{l-1}\log(t,\tau)^{-1}\ \mathcal{F}_{s,l}(t,\tau)\ , & b\,t\geq\delta'\tau_j\\ t^{-1}\ (\frac{t}{\tau_j})^{1/2}\ \mathcal{P}_{l-1}\log(t,\tau)^{-1}\ \mathcal{F}_{s,l}(t,\tau)\ , & b\,t<\delta'\tau_j\end{array}\right. \tag{105}$$

[18] remember that the vertex x_s in $T_{l_2}^{s_2+1})$ is a root vertex by construction

$$|\partial_t \mathcal{L}^{\varepsilon,t}_{2,l}(x_1, \varphi^{(2)}_{\tau,y_2})| \leq \begin{cases} t^{-2}\ \mathcal{P}_{l-1} \log t^{-1}\ \mathcal{F}_{2,l}(t,\tau)\ , & b\,t \geq \delta'\tau_2 \\ t^{-2}\ (\frac{t}{\tau})^{1/2}\ \mathcal{P}_{l-1} \log t^{-1}\ \mathcal{F}_{2,l}(t,\tau)\ , & b\,t < \delta'\tau_2\ . \end{cases} \tag{106}$$

D) From the bounds on the derivatives $\partial_t \mathcal{L}^{\varepsilon,t}_{n,l}$ we then verify the induction hypothesis on integrating over t. In all cases we need the bound

$$\mathcal{F}_{s,l}(t',\tau) \leq \mathcal{F}_{s,l}(t,\tau) \quad \text{for} \quad t' \leq t\ , \tag{107}$$

which follows directly from the definition (46).

a) In the cases $\underline{n+r>4}$ we have, due to the boundary conditions encoded in the form of (25)

$$\mathcal{L}^{\varepsilon,t}_{n,l}(x_1,\varphi) = \int_\varepsilon^t dt'\ \partial_{t'} \mathcal{L}^{\varepsilon,t'}_{n,l}(x_1,\varphi)\ .$$

Then we get from (87), (91)-(93), due to (107),

$$|\,\mathcal{L}^{\varepsilon,t}_{n,l}(x_1,\varphi_s)| \leq t^{\frac{n-4}{2}}\ \mathcal{P}_l \log(t,\tau)^{-1}\ \mathcal{F}_{s,l}(t,\tau) \tag{108}$$

$$|\mathcal{L}^{\varepsilon,t}_{n,l}(x_1, E^{(r)}_{(k)}\,\varphi_s)| \leq |\,\omega^{(r)}(x_1)|\, t^{\frac{n+r-4}{2}}\ \mathcal{P}_l \log(t,\tau)^{-1}_k\ \mathcal{F}_{s,l}(t,\tau)\ , \quad n>4 \tag{109}$$

$$|\mathcal{L}^{\varepsilon,t}_{4,l}(x_1, E^{(r)}_{(k)}\,\varphi_s)| \leq |\,\omega^{(r)}(x_1)|\, t^{\frac{r}{2}}\ \mathcal{P}_{l-1} \log(t,\tau)^{-1}_k\ \mathcal{F}_{s,l}(t,\tau)\ , \quad r>0 \tag{110}$$

$$|\mathcal{L}^{\varepsilon,t}_{2,l}(x_1, E^{(3)}_{(2)}\varphi_s)| \leq |\,\omega^{(3)}(x_1)|\, t^{\frac{1}{2}}\ \mathcal{P}_{l-1} \log t^{-1}\ \mathcal{F}_{s,l}(t,\tau)\ , \tag{111}$$

which proves the proposition for these cases.

b) Similarly, for $\underline{n \geq 4}$ the boundary conditions (25) imply that

$$\mathcal{L}^{\varepsilon,t}_{n,l}(x_1,\varphi^{(j)}_s) = \int_\varepsilon^t dt'\ \partial_{t'} \mathcal{L}^{\varepsilon,t'}_{n,l}(x_1,\varphi^{(j)}_s)\ ,$$

and we then obtain from (104), (105) together with (107)

$$|\mathcal{L}^{\varepsilon,t}_{n,l}(x_1,\varphi^{(j)}_s)| \leq t^{\frac{n-4}{2}} \left(\frac{t}{\tau_j}\right)^{1/2} \mathcal{P}_l \log(t,\tau)^{-1}\ \mathcal{F}_{s,l}(t,\tau)\ . \tag{112}$$

We note that for $b\,t \geq \delta'\tau_j$ the integral $\int_\varepsilon^t dt'$ has to be split into $\int_\varepsilon^{\delta'\tau_j/b} dt' + \int_{\delta'\tau_j/b}^t dt'$, and that in the case $n=4$ the polynomial in logarithms may increase in degree by one unit due to the logarithmically divergent t-integral, see (115)-(117) below for more details.

c) In the case $\underline{n=4\,,\ r=0}$ we start from the decomposition (53)

$$\mathcal{L}^{\varepsilon,t}_{4,l}(x_1,\varphi) = c^{\varepsilon,t}_l(x_1)\ \varphi(x_1,x_1,x_1) + \ell^{\varepsilon,t}_{4,l}(x_1,\varphi)\ .$$

On integrating the bound for $c_l^{\varepsilon,t}(x_1)$, (95), from t to 1 and using the boundary condition (68) we get

$$| \, c_l^{\varepsilon,t}(x_1) \, | \leq \mathcal{P}_l \log t^{-1} \, . \tag{113}$$

Taking together (95) and (88) we verify

$$|\partial_t \ell_{4,l}^{\varepsilon,t}(x_1,\varphi_s)| \;\leq\; t^{-1} \, \mathcal{P}_{l-1} \log(t,\tau)^{-1} \, \mathcal{F}_{s,l}(t,\tau) \; .$$

A sharper bound for $\partial_t \ell_{4,l}^{\varepsilon,t}(x_1,\varphi_s)$, which when integrated over $t \geq \varepsilon$ stays uniformly bounded in ε, is obtained as follows. In the case $s = 4$ [19] we decompose the test function

$$\varphi_4(x_2,x_3,x_4) := \prod_{i=2}^{4} K(\tau_i,x_i,y_i) = \varphi_4(x_1,x_1,x_1) + \psi(x_2,x_3,x_4) \, ,$$

$$\psi(x_2,x_3,x_4) = \sum_{i=2}^{4} \prod_{f=2}^{i-1} K(\tau_f,x_1,y_f) \, K^{(1)}(\tau_i,x_i,x_1;y_i) \prod_{j=i+1}^{4} K(\tau_j,x_j,y_j) \; .$$

Then $\ell_{4,l}^{\varepsilon,t}(x_1,\varphi_4) = \mathcal{L}_{4,l}^{\varepsilon,t}(x_1,\psi)$, and hence the FE (32) provides

$$\begin{aligned} \partial_t \, \ell_{4,l}^{\varepsilon,t}(x_1,\varphi_4) \;=\; & \frac{1}{2} \int_{x_2,x_3,x_4,x,y} \psi(x_2,x_3,x_4) \, C_t(x,y) \, \Big\{ \mathcal{L}_{6,l-1}^{\varepsilon,t}(x_1,\ldots,x_4,x,y) \\ & - \sum_{l_1+l_2=l} \Big[\mathcal{L}_{4,l_1}^{\varepsilon,t}(x_1,x_2,x_3,x) \, \mathcal{L}_{2,l_2}^{\varepsilon,t}(y,x_4) \; + \mathcal{L}_{2,l_1}^{\varepsilon,t}(x_1,x) \; \mathcal{L}_{4,l_2}^{\varepsilon,t}(y,x_2,x_3,x_4) \Big]_{sym} \Big\} \; . \end{aligned} \tag{114}$$

The r.h.s. is a sum over expressions of the same form as the one for $\partial_t \mathcal{L}_{4,l}^{\varepsilon,t}(x_1,\varphi^{(j)}_{\tau_{2,s},y_{2,s}})$ in part C. Setting $\tau = \inf_j\{\tau_j\}$ we obtain, in the same way as there, the bound

$$|\partial_t \ell_{4,l}^{\varepsilon,t}(x_1,\varphi_s)| \;\leq\; \begin{cases} t^{-1} \, \mathcal{P}_{l-1} \log(t,\tau)^{-1} \, \mathcal{F}_{s,l}(t,\tau) \; , & b\,t \geq \delta'\tau \; , \\ t^{-1} \, (\frac{t}{\tau})^{1/2} \, \mathcal{P}_{l-1} \log(t,\tau)^{-1} \, \mathcal{F}_{s,l}(t,\tau) \; , & b\,t < \delta'\tau \; . \end{cases} \tag{115}$$

Using the boundary condition (66) we integrate (115) over t. This gives for $b\,t < \delta'\tau$ (and ε sufficiently small)

$$\Big| \int_\varepsilon^t dt' \; \partial_{t'} \, \ell_{4,l}^{\varepsilon,t'}(x_1,\varphi_s) \Big| \;\leq\; (\frac{t}{\tau})^{1/2} \, \mathcal{P}_{l-1} \log t^{-1} \, \mathcal{F}_{s,l}(t,\tau) \, , \tag{116}$$

and for $b\,t > \delta'\tau$, in which case we may have $t > \tau$ or $t < \tau$,

$$\Big| \int_\varepsilon^t dt' \; \partial_{t'} \, \ell_{4,l}^{\varepsilon,t'}(x_1,\varphi_s) \Big| \;\leq\; \Big| \int_\varepsilon^{\delta'\tau/b} dt' \partial_{t'} \, \ell_{4,l}^{\varepsilon,t'}(x_1,\varphi_s) \Big| + \Big| \int_{\delta'\tau/b}^t dt' \partial_{t'} \, \ell_{4,l}^{\varepsilon,t'}(x_1,\varphi_s) \Big|$$

[19] In this case $\varphi_i(x_i) = K(\tau_i,x_i,y_i)$, $1 \leq i \leq 4$. If $\varphi_i = \mathbf{1}$ for some i, the corresponding contribution to the subsequent sum over i vanishes.

$$\leq \left(\left(\frac{\delta'}{b}\right)^{1/2} \mathcal{P}_{l-1} \log \frac{b}{\delta'\tau} + \mathcal{P}_l \log \frac{b}{\delta'\tau} \right) \mathcal{F}_{s,l}(t,\tau) \ . \tag{117}$$

Hence, absorbing powers of $\log(\delta'/b)$ in the coefficients of $\mathcal{P}_l \log$ as usual,

$$|\ell^{\varepsilon,t}_{4,l}(x_1,\varphi_s)| \leq \mathcal{P}_l \log \tau^{-1} \mathcal{F}_{s,l}(t,\tau) \ . \tag{118}$$

From (53), (113), (116) and (118) we obtain

$$|\mathcal{L}^{\varepsilon,t}_{4,l}(x_1,\varphi_s)| \leq \mathcal{P}_l \log(t,\tau)^{-1} \mathcal{F}_{s,l}(t,\tau) \ .$$

d) In the case $\underline{n=2}$ we have the decomposition (52). In addition to the bounds (95)-(97) to be integrated from 1 to $t \leq 1$, we need for $\partial_t \ell^{\varepsilon,t}_{2,l}(x_1,\varphi)$, (56), a bound, which upon integration from ε to t becomes a uniformly bounded function on $\varepsilon \geq 0$. To this end we use the form (58) and choose the test function $\varphi(x_2) = K(\tau, x_2, y_2)$. Taking into account the bound (94) for $n=2, r=0$ together with (A.31) yields

$$|\partial_t \ell^{\varepsilon,t}_{2,l}(x_1,\varphi)| \leq \int_{x_2} t^{-\frac{1}{2}} \mathcal{P}_{l-1} \log t^{-1} \mathcal{F}^{(12)}_{2,l}(t) \int_0^1 d\rho \frac{(1-\rho)^2}{2!} |\nabla^3_{(1)} K(\tau, X(\rho), y_2)|$$

$$\leq t^{-\frac{1}{2}} \tau^{-\frac{3}{2}} \mathcal{P}_{l-1} \log t^{-1} \int_{x_2} \mathcal{F}^{(12)}_{2,l}(t) \int_0^1 d\rho\, K(\tau_{\delta'}, X(\rho), y_2)$$

where (15) has been used. By definition we have

$$\mathcal{F}^{(12)}_{2,l}(t) = \mathcal{F}_{2,l}(t;x_1,x_2) = \sum_{T^{2,(12)}_l} \mathcal{F}_{2,l}(t;T^{2,(12)}_l;x_1,x_2)$$

$$= \sum_{n=1}^{3l-2} \sup_{\{t_{I_\nu}|\varepsilon \leq t_{I_i} \leq t,\, i=1\cdots,n\}} [\prod_{1\leq\nu\leq n} \int_{z_\nu}] C_{t_{I_1},\delta}(x_1,z_1)\ldots C_{t_{I_n},\delta}(z_n,x_2)$$

$$= \sum_{n=1}^{3l-2} \sup_{\{t_{I_\nu}|\varepsilon \leq t_{I_\nu} \leq t,\, \nu=1\cdots,n\}} C_{\sum_1^n t_{I_\nu},\delta}(x_1,x_2)$$

where we used (5) and (51). We then proceed similarly as in and after (101). Setting $N = 3l-2$ we bound for $Nbt < \delta'\tau$ and for $n \leq N$ as in (102)

$$C_{\sum_1^n t_{I_\nu},\delta}(x_1,x_2) \int_0^1 d\rho\ K(\tau_{\delta'}, X(\rho), y_2) \leq O(1)\ C_{2\sum_1^n t_{I_\nu},\delta}(x_1,x_2)\ K(\tau_\delta, x_1, y_2) \tag{119}$$

so that, observing (4),

$$|\partial_t \ell^{\varepsilon,t}_{2,l}(x_1,\varphi)| \leq \tau^{-3/2}\, t^{-1/2}\, \mathcal{P}_{l-1} \log t^{-1}\ K(\tau_\delta, x_1, y_2)\ , \quad Nbt < \delta'\tau \ . \tag{120}$$

To verify the induction hypothesis (73) we resort to the decomposition (52) and denote the sum of the first, second and third terms there by $\mathcal{L}^{\varepsilon,t}_{2,l}(x_1,\,\varphi)_{rel}$. Integrating the corresponding bounds (95)-(97) from 1 to t, and using again (15) gives

$$|\,\mathcal{L}^{\varepsilon,t}_{2,l}(x_1,\,\varphi)_{rel}| < \Big(\frac{1}{t}\;\mathcal{P}_{l-1}\log t^{-1}\;+\frac{1}{(t\tau)^{\frac{1}{2}}}\mathcal{P}_{l-2}\log t^{-1}+\frac{1}{\tau}\mathcal{P}_{l-1}\log t^{-1}\Big)\;K(\tau_\delta,x_1,y_2)\;. \tag{121}$$

Integrating the remainder (120) from (small) ε with vanishing initial condition (66) to $t<\delta'\tau/(bN)$ leads to

$$|\,\ell^{\varepsilon,t}_{2,l}\;(x_1,\varphi)|\;\le\;\tau^{-3/2}\;t^{1/2}\;\mathcal{P}_{l-1}\log t^{-1}\;K(\tau_\delta,x_1,y_2)\;. \tag{122}$$

By way of (52) we obtain from the bounds (88) and (95)-(97) the bound for $Nb\,t\ge\delta'\tau$

$$|\,\partial_t\,\ell^{\varepsilon,t}_{2,l}\;(x_1,\varphi)|\le\;t^{-2}\;\mathcal{P}_{l-1}\log(t,\tau)^{-1}\;\mathcal{F}_{2,l}(t,\tau)\;+ \tag{123}$$

$$\Big(t^{-2}\;\mathcal{P}_{l-1}\log t^{-1}\;+\sum_{j=1}^{2}t^{\frac{j-4}{2}}\;\tau^{-j/2}\;\mathcal{P}_{l-2}\log t^{-1}\Big)\;K(\tau_\delta,x_1,y_2)\;.$$

Hence, integration and majorization, again observing both $\tau>t$ and $t<\tau$, gives for $t>\delta'\tau/(bN)$

$$|\,\ell^{\varepsilon,t}_{2,l}\;(x_1,\varphi)|\;\le\;\Big(\frac{bN}{\delta'\tau}\,\mathcal{P}_{l-1}\log\frac{bN}{\delta'\tau}+\theta(t-\tau)\frac{1}{\tau}\mathcal{P}_{l-1}\log\tau^{-1}\Big)\,\mathcal{F}_{2,l}(t,\tau)\;+ \tag{124}$$

$$\Big(\frac{bN}{\delta'\tau}\,\mathcal{P}_{l-1}\log\frac{bN}{\delta'\tau}+\Big(\frac{bN}{\delta'\tau^2}\Big)^{1/2}\mathcal{P}_{l-2}\log\frac{bN}{\delta'\tau}\;+\frac{1}{\tau}\mathcal{P}_{l-1}\log\frac{bN}{\delta'\tau}\Big)\,K(\tau_\delta,x_1,y_2)\;.$$

From (121), (122) and (124), absorbing constants as usual in $\mathcal{P}\log$, we then get [20]

$$|\,\mathcal{L}^{\varepsilon,t}_{2,l}(x_1,\,\varphi)\,|\;\le\;(t,\tau)^{-1}\;\mathcal{P}_{l-1}\log(t,\tau)^{-1}\;\mathcal{F}_{2,l}(t,\tau) \tag{125}$$

in accord with (73).

[20] The bound (124) diverges linearly with δ', whereas in (117) the divergence was only logarithmic. This indicates rapid growth since the bounds then behave as $(\delta')^{-l}$, a factor of $(\delta')^{-1}$ being produced per loop order. Without trying at all to optimize constants, we still note that it is possible to choose for this case $\delta=2$ in $K(\tau_\delta,x_1,y_2)$ and bound the two point function inductively by $\mathcal{F}_{2,l}(t,3\tau)$ without changing the bounds on the other functions. The only place in the proof where there is a modification due to this factor is in part A2). But here the value of τ appearing is $t/2$, see (86), and $3t/2$ can be accommodated for in the proof by introducing a new vertex of incidence number 2 while respecting the bound on the number of those vertices. A value $\delta=2$ then gives for suitable choice of b the value $b/\delta'\simeq 6$.

To establish the bounds on $\mathcal{L}^{\varepsilon,t}_{2,l}(x_1, E^{(r)}_{(2)}\varphi)$, $r = 1, 2$, we expand the respective test functions as follows, employing (A.30) and using the notations (54), (41)

$$\mathcal{L}^{\varepsilon,t}_{2,l}(x_1, E^{(1)}_{(2)}\varphi) = \varphi(x_1) f^{\mu,\varepsilon,t}_l(x_1)\ \omega_\mu(x_1) + 2\, b^{\mu\nu,\varepsilon,t}_l(x_1)\, \omega_\mu(x_1)\, (\nabla_\nu\varphi)(x_1)$$

$$+ \int_{x_2} F^{(1)}_{(12)} \mathcal{L}^{\varepsilon,t}_{2,l}(x_1, x_2) \int_0^1 d\rho\, \frac{(1-\rho)}{d^2(x_1,x_2)}\, \dot{X}^\mu(\rho)\, \dot{X}^\nu(\rho)(\nabla_\mu\nabla_\nu\varphi)(X(\rho))\,, \tag{126}$$

$$\mathcal{L}^{\varepsilon,t}_{2,l}(x_1, E^{(2)}_{(2)}\varphi) = -2\, \varphi(x_1)\, b^{\mu\nu,\varepsilon,t}_l(x_1)\, \omega^{(2)}_{\mu\nu}(x_1)$$

$$+ \int_{x_2} F^{(2)}_{(12)} \mathcal{L}^{\varepsilon,t}_{2,l}(x_1, x_2) \int_0^1 d\rho\, \frac{1}{d(x_1,x_2)}\, \dot{X}^\mu(\rho)\, (\nabla_\mu\varphi)(X(\rho))\,. \tag{127}$$

The local, i.e. relevant terms have already been dealt with in (96), (97), and the remainders are treated as $\ell^{\varepsilon,t}_{2,l}\ (x_1, \varphi)$; one obtains

$$|\, \mathcal{L}^{\varepsilon,t}_{2,l}(x_1, E^{(1)}_{(2)}\varphi_2)| \ \le\ |\, \omega^{(1)}(x_1)|\ (t,\tau)^{-1/2}\ \mathcal{P}_{l-1} \log(t,\tau)^{-1}\ \mathcal{F}_{2,l}(t,\tau)\ ,$$

$$|\, \mathcal{L}^{\varepsilon,t}_{2,l}(x_1, E^{(2)}_{(2)}\varphi_2)| \ \le\ |\omega^{(2)}(x_1)|\ \mathcal{P}_{l-1} \log(t,\tau)^{-1}\ \mathcal{F}_{2,l}(t,\tau)\ .$$

Finally, we realize that $\mathcal{L}^{\varepsilon,t}_{2,l}(x_1, \varphi^{(2)}_2)$ equals the r.h.s. of (52) without its first term. Proceeding again similarly as before - see (121), (122) and (124) - provides

$$|\mathcal{L}^{\varepsilon,t}_{2,l}(x_1, \varphi^{(2)}_2)| \ \le\ (\frac{t}{\tau})^{1/2}\ (t,\tau)^{-1}\ \mathcal{P}_{l-1} \log(t,\tau)^{-1}\ \mathcal{F}_{2,l}(t,\tau)\ .$$

This ends the proof of Proposition 1. ∎

The behaviour of the CAS upon removing the UV cutoff, i.e. $\varepsilon \searrow 0$, follows from
Proposition 2:
Let ε be (sufficiently) small. With the notations, conventions and the same class of renormalization conditions as in Proposition 1 we have the bounds

$$|\, \partial_\varepsilon\, \mathcal{L}^{\varepsilon,t}_{n,l}(x_1, \varphi_{\tau_{2,s},y_{2,s}})| \ \le\ \varepsilon^{-\frac{1}{2}}\ \mathcal{P}_l \log\varepsilon^{-1}\ t^{\frac{n-5}{2}}\ \mathcal{F}_{s,l}(t,\tau) \tag{128}$$

$$|\, \partial_\varepsilon\, \mathcal{L}^{\varepsilon,t}_{n,l}(x_1, E^{(r)}_{(i)}\varphi_{\tau_{2,s},y_{2,s}})| \ \le\ \varepsilon^{-\frac{1}{2}}\ \mathcal{P}_l \log\varepsilon^{-1}\ |\, \omega^{(r)}(x_1)|\, t^{\frac{n+r-5}{2}}\ \mathcal{F}_{s,l}(t,\tau) \tag{129}$$

$$|\, \partial_\varepsilon\, \mathcal{L}^{\varepsilon,t}_{n,l}(x_1, \varphi^{(j)}_{\tau_{2,s},y_{2,s}})| \ \le\ \varepsilon^{-\frac{1}{2}}\ \mathcal{P}_l \log\varepsilon^{-1}\ t^{\frac{n-4}{2}}\ \tau_j^{-\frac{1}{2}}\ \mathcal{F}_{s,l}(t,\tau) \tag{130}$$

$$|\, \partial_\varepsilon\, F^{(0)}_{(12)}\mathcal{L}^{\varepsilon,t}_{n,l}(x_1, x_2, \varphi_{\tau_{2,s},y_{2,s}})| \ \le\ \varepsilon^{-\frac{1}{2}}\ \mathcal{P}_l \log\varepsilon^{-1}\ t^{\frac{n-2}{2}}\ \mathcal{F}^{(12)}_{s,l}(t,\tau) \tag{131}$$

$$|\, \partial_\varepsilon\, \mathcal{L}^{\varepsilon,t}_{2,l}(x_1, \varphi_{\tau,y})| \ \le\ \varepsilon^{-\frac{1}{2}}\ \mathcal{P}_{l-1} \log\varepsilon^{-1}\ (t,\tau)^{-\frac{3}{2}}\ \mathcal{F}_{2,l}(t,\tau) \tag{132}$$

$$|\, \partial_\varepsilon\, F^{(0)}_{(12)}\mathcal{L}^{\varepsilon,t}_{2,l}(x_1, x_2)| \ \le\ \varepsilon^{-\frac{1}{2}}\ \mathcal{P}_{l-1} \log\varepsilon^{-1}\ \mathcal{F}^{(12)}_{2,l}(t)\ . \tag{133}$$

Proof: We apply the method developed in the previous proof. The bound (128) obviously holds in the starting case $n=4, l=0$. Because of the bare interaction (25) the FE (33) is used if $n+r>4$, where the difference test function in (130) and the modified insertion in (131),(133) count as $r=1$ and $r=3$, respectively. Regarding the r.h.s. of (33) we note that the first and second term do not contribute to the cases considered, and the third one only if $n=2,4,6$.

Proceeding inductively as in A, B) and C) of the previous proof, and using the bounds of Proposition 1, reproduces (128) for $n>4$ and (129)-(131) and (133).

The FE (34) provides bounds on the relevant parts of the cases $n+r \le 4$. As the renormalization conditions (67), (68), (69) depend at most weakly on ε, we obtain inductively

$$|\,\partial_\varepsilon\, c_l^{\varepsilon,t}(x_1)\,| \le \varepsilon^{-\frac{1}{2}}\ \mathcal{P}_{l-1}\log\varepsilon^{-1}\ \cdot t^{-\frac{1}{2}} \quad , \quad |\,\partial_\varepsilon\, a_l^{\varepsilon,t}(x_1)\,| \le \varepsilon^{-\frac{1}{2}}\,\mathcal{P}_{l-1}\log\varepsilon^{-1}\cdot t^{-\frac{3}{2}} \tag{134}$$

$$|\,\partial_\varepsilon\, f_l^{\mu,\varepsilon,t}(x_1)\,\omega_\mu(x_1)| \ \le\ |\,\omega(x_1)|\ \varepsilon^{-\frac{1}{2}}\ \mathcal{P}_{l-1}\log\varepsilon^{-1}\ \cdot t^{-1} \tag{135}$$

$$|\,\partial_\varepsilon\, b_l^{\mu\nu,\varepsilon,t}(x_1)\,\omega^{(2)}_{\mu\nu}(x_1)\ | \ \le\ |\,\omega^{(2)}(x_1)\,|\ \varepsilon^{-\frac{1}{2}}\ \mathcal{P}_{l-1}\log\varepsilon^{-1}\ \cdot t^{-\frac{1}{2}}\ . \tag{136}$$

With the aid of the decomposition (53), the bound (128) for $n=4$ follows from (134) and (130). It remains to show (132). We use the decomposition (52) and perform similar steps as in D)d). From (58) and (133) we obtain

$$\begin{aligned}|\,\partial_\varepsilon\,\ell_{2,l}^{\varepsilon,t}\,(x_1,\varphi_{\tau,y})| \ &\le\ \int_{x_2}|\,\partial_\varepsilon\, F^{(0)}_{(12)}\mathcal{L}_{2,l}^{\varepsilon,t}(x_1,x_2)\,|\ \int_0^1 d\rho\,\frac{(1-\rho)^2}{2!}\,|\,(\nabla^{\,3}\varphi_{\tau,y})(X(\rho))\,| \\ &\le\ \varepsilon^{-\frac{1}{2}}\ \mathcal{P}_{l-1}\log\varepsilon^{-1}\ \tau^{-\frac{3}{2}}\int_{x_2}\mathcal{F}^{(12)}_{2,l}(t)\int_0^1 d\rho\, K(\tau_{\delta'},X(\rho),y)\end{aligned}$$

and herefrom, cf. (51), (119) for $Nb\,t<\delta'\tau$ $(N=3l-2)$,

$$|\,\partial_\varepsilon\,\ell_{2,l}^{\varepsilon,t}\,(x_1,\varphi_{\tau,y})|\ \le\ \varepsilon^{-\frac{1}{2}}\ \mathcal{P}_{l-1}\log\varepsilon^{-1}\ \tau^{-3/2}\ K(\tau_\delta,x_1,y)\ . \tag{137}$$

From (134)-(136) follows

$$|\,\partial_\varepsilon\,\mathcal{L}_{2,l}^{\varepsilon,t}(x_1,\,\varphi_{\tau,y})_{rel}| < \varepsilon^{-\frac{1}{2}}\ \mathcal{P}_{l-1}\log\varepsilon^{-1}\ \cdot\frac{1}{t^{\frac{1}{2}}}\left(\frac{1}{t}+\frac{1}{(t\tau)^{\frac{1}{2}}}+\frac{1}{\tau}\right)\,K(\tau_\delta,x_1,y)\,. \tag{138}$$

On account of (52) the bounds (137), (138) establish (132) for $Nb\,t<\delta'\tau$.

To obtain an extension of the bound (137) to $Nb\,t\ge\delta'\tau$ we again resort to the decomposition (52), yielding

$$\begin{aligned}\partial_t\,\partial_\varepsilon\,\ell_{2,\,l}^{\varepsilon,t}(x_1,\varphi)\ &=\ \partial_t\,\partial_\varepsilon\,\mathcal{L}_{2,\,l}^{\varepsilon,t}(x_1,\,\varphi) \\ -\ \partial_t\,\partial_\varepsilon\, a_l^{\varepsilon,t}(x_1)\varphi(x_1)\, &+\ \partial_t\,\partial_\varepsilon\, f_l^{\mu,\varepsilon,t}(x_1)\,\omega_\mu(x_1)\,+\ \partial_t\,\partial_\varepsilon\, b_l^{\mu\nu,\varepsilon,t}(x_1)\,\omega^{(2)}_{\mu\nu}(x_1)\,,\end{aligned} \tag{139}$$

with $\omega_\mu(x)=\nabla_\mu\varphi(x)$, $\omega^{(2)}_{\mu\nu}(x)=\nabla_\mu\nabla_\nu\varphi(x)$, $\varphi(x)=K(\tau,x,y)$. Employing on the r.h.s. of (139) in the various terms the corresponding FE (32) derived w.r.t. ε and then making

use of bounds of Proposition 1 and of Proposition 2 already established inductively, leads with now familiar steps to

$$|\,\partial_t\,\partial_\varepsilon\,\ell^{\varepsilon,t}_{2,l}(x_1,\varphi)|\le\ \varepsilon^{-\frac12}\,\mathcal{P}_{l-1}\log\varepsilon^{-1}\ \cdot\Big[\,(t,\tau)^{-\frac52}\ \mathcal{F}_{2,l}(t,\tau)$$

$$+\ \Big(t^{-\frac52}+\tau^{-\frac12}\,t^{-2}+\tau^{-1}\,t^{-\frac32}\Big)\,K(\tau_\delta,x_1,y)\Big]\,. \tag{140}$$

On integrating $\partial_t\,\partial_\varepsilon\,\ell^{\varepsilon,t}_{2,l}\ (x_1,\varphi)$ from $t=\varepsilon$ (small) with vanishing initial condition up to $t\ge\delta'\tau/bN$ the integral has to be split at $t=\delta'\tau/bN$. A bound on the lower part of the integral is given by (137). The upper part of the integral can be bounded using (140) observing both $\tau>t$ and $\tau<t$, and majorizing constants. Combining both contributions yields for $t\ge\delta'\tau/bN$

$$|\,\partial_\varepsilon\,\ell^{\varepsilon,t}_{2,l}(x_1,\varphi)|\le\ \varepsilon^{-\frac12}\,\mathcal{P}_{l-1}\log\varepsilon^{-1}\ \cdot\ \Big(\frac{bN}{\delta'\tau}\Big)^{\frac32}$$

$$\cdot\ \Big[\,\mathcal{F}_{2,l}(t,\tau)+\Big(1+\Big(\frac{\delta'}{bN}\Big)^{\frac12}+\frac{\delta'}{bN}\Big)\,K(\tau_\delta,x_1,y)\Big]\,. \tag{141}$$

Taking into account once more the decomposition (52), the bound (138) on the relevant part together with the bounds (137), (141) on the remainder reproduce (132). Thus the proof of Proposition 2 is complete. ∎

From (128), (132) follows the integrability at $\varepsilon=0$ and hence the existence of finite limits

$$\lim_{\varepsilon\searrow 0}\ \mathcal{L}^{\varepsilon,t}_{n,l}\,(x_1,\varphi_{\tau_{2,s},y_{2,s}})\,,\qquad n\ge 2\ .$$

Proposition 3:
With the notations, conventions and the same class of renormalization conditions as in Proposition 1 - up to the fact that the constants in $\mathcal{P}_l\log$ may now also depend on the mass m - we claim the following bounds for the CAS in the interval $1\le t\le\infty$:

$$|\,\mathcal{L}^{\varepsilon,t}_{n,l}(x_1,\varphi_{\tau,y_{2,s}})|\ \le\ \mathcal{P}_l\log\tau^{-1}\ \mathcal{F}^{\,t}_{s,l}(\tau)\ ,\quad n\ge 4 \tag{142}$$

$$|\,\mathcal{L}^{\varepsilon,t}_{2,l}(x_1,\varphi_{\tau,y})|\ \le\ (1,\tau)^{-1}\ \mathcal{P}_{l-1}\log(1,\tau)^{-1}\ \mathcal{F}^{\,t}_{2,l}(\tau)\ . \tag{143}$$

The definition of $\mathcal{F}^{\,t}_{s,l}(\tau)$ is given in (50).

Proof : The bounds stated in the proposition are proven inductively using again the standard scheme. The boundary conditions are the bounds from Proposition 1 taken at $t=1$. They obviously satisfy the bounds (142), (143). The FE is treated in the same

way as in parts A1) and A2) of the proof of Proposition 1. The integration w.r.t t is performed using the fact that $\mathcal{F}^t_{s,l}(\tau)$ is montonically increasing with t. As regards part A1) we now use for $t \geq 1$ instead of (80) now $C_t(z,z') \leq O(1)\,\exp(-(m^2-\delta)\,t)$, which results from the upper bounds (6), (8) on the heat kernel, and obtain upon integration

$$\int_1^t dt'\;\mathcal{F}^{t'}_{s,l}(\tau)\;e^{-(m^2-\delta)\,t'}\;\leq\;O(1/m^2)\,\mathcal{F}^t_{s,l}(\tau)\;.$$

As regards A2) the internal line generated, which connects the two (partial) trees, see (84), (85), has the weight (86). Integrating, we majorize the weights of the other internal lines by their values at t and use for (86)

$$\int_1^t dt'\,C_{t'}(z',z'')\;\leq\;C_{\underline{t}}\,(z',z'')+\int_1^t dt'\,C_{t'}(z',z'')$$

valid for any $0<\underline{t}\leq 1$, thus reproducing the weight factor $\mathcal{F}^t_{s,l}(\tau)$, (50), in this case, too. ∎

Note that the renormalization conditions at $t=1$ are in one to one relation with the values of the corresponding relevant terms at $t=\infty$, which have been shown to be finite for $m^2>0$ in according to Proposition 3. Therefore renormalization conditions at $t=1$ are tantamount to renormalization conditions at $t=\infty$.

We want to close this section with some comments on the test functions considered and on possible extensions of the class of test functions. We stay with some informal remarks here, we did not rigorously analyse the problem of what is a "natural large" class of test functions. First note that our test functions can be arbitrarily well localized around any point of the manifold. This is an essential criterion for their viability from the physical point of view. Secondly the class of test functions can be extended by linearity (36). Since our bounds are in terms of the weight factors decaying with the tree distance between the points $x_1, y_2, \ldots, y_s$ it is quite evident that the functionals $\mathcal{L}^{\varepsilon,t}_{n,l}(x_1,\varphi)$ can also be extended continuously by bounded convergence to test functions which are infinite sums $\sum_i \lambda_i\;\varphi_{i,\tau^{(i)}_{2,s},y^{(i)}_{2,s}}$ with $\sum|\lambda_i|<\infty$. To go further one could either prove (in a more functional analysis type of approach) that our test functions are dense e.g. in the set of smooth rapidly decaying functions on $\mathcal{M}$ w.r.t. a suitable norm, and that the $\mathcal{L}^{\varepsilon,t}_{n,l}(x_1,\varphi)$ are continuous w.r.t. this norm. Or one could try to directly extend the previous proof to more general test functions in a second step. In this case the crucial part would be to maintain the line of argument presented in part C), (99) to (103), of the previous proof.

7 Scaling transformations and the minimal form of the bare action

In this section we want to show that the theory can be renormalized starting from a bare (inter)action of the form (21). This requires that we do not introduce any position dependent quantity in the theory which is not intrinsic to $(\mathcal{M}, g)$. Thus we only consider position independent coupling λ, and renormalization conditions in terms of intrinsic geometric quantities. We then introduce scaling tranformations of the following kind :
For a four-dimensional Riemannian manifold $(\mathcal{M}, g)$ we scale its metric by a constant conformal factor, [NePa],

$$\rho \in \mathbf{R}_+ : \qquad g_{\mu\nu}(x) \to \rho^2\, g_{\mu\nu}(x)\,, \quad \text{shortly} \;\; g \to \rho^2\, g\,. \tag{144}$$

This leads to corresponding changes of geometrical quantities

$$\begin{gathered} g^{\mu\nu} \to \rho^{-2} g^{\mu\nu}, \quad \Delta \to \rho^{-2}\Delta, \quad |\,g|^{1/2} \to \rho^4 |\,g|^{1/2}, \quad \tilde{\delta} \to \rho^{-4}\,\tilde{\delta} \\ d(x,y) \to \rho\, d(x,y), \quad \sigma(x,y)^\mu \to \sigma(x,y)^\mu \\ \Gamma^\lambda_{\mu\nu} \to \Gamma^\lambda_{\mu\nu}\,, \quad \nabla_\mu \to \nabla_\mu\,, \quad R^\lambda_{\;\mu\nu\sigma} \to R^\lambda_{\;\mu\nu\sigma}\,, \quad R_{\mu\nu} \to R_{\mu\nu}\,, \quad R \to \rho^{-2} R\,. \end{gathered} \tag{145}$$

Moreover, the heat kernel $K(t,x,y\,;g)$ satisfies the scaling relation

$$K(t,x,y\,;\,g) \;=\; \rho^4\, K(\,\rho^2 t, x, y\,;\,\rho^2 g)\,, \tag{146}$$

which follows from its evolution equation $(\partial_t - \Delta_g)K(t,x,y\,;\,g) \;=\; 0$ together with stochastic completeness (4). As a consequence the regularized free propagator (18), $0 < \varepsilon < t \le \infty$,

$$C^{\,\varepsilon,t}(x,y;\,m^2\,,g) \;=\; \int_\varepsilon^t dt'\, e^{\,-m^2 t'}\, K(t',x,y;\,g)\,,$$

satisfies

$$C^{\,\varepsilon,t}(x,y;m^2\,,g) \;= \rho^2\, C^{\,\rho^2\varepsilon,\,\rho^2 t}(x,y;\frac{m^2}{\rho^2}\,,\rho^2 g)\,. \tag{147}$$

Regarding for a moment the action of the *classical scalar field theory*,

$$S(\varphi, m^2, \xi, \lambda; g) \;= \frac{1}{2}\int_x \Big(\varphi(-\Delta)\varphi + m^2\varphi^2 + \xi\, R(x)\,\varphi^2 + 2\frac{\lambda}{4!}\,\varphi^4\Big)\,, \tag{148}$$

we observe, that it is invariant if we supplement the scaling (144) of the metric by the transformations

$$\varphi(x) \to \rho^{-1}\varphi(x)\,, \quad m^2 \to \rho^{-2} m^2\,, \quad \xi \to \xi\,, \quad \lambda \to \lambda\,. \tag{149}$$

We now consider the perturbative expansion of a regularized $\lambda\phi^4$- theory without counter terms, i.e. in (20) we have $L^{\varepsilon,\varepsilon}(\phi) = \lambda \int dV(x)\, \phi^4(x)$. A Feynman diagram contributing to an n-point CAS having v four-vertices and I internal lines obeys the topological relation $4v = n + 2I$. This together with the scaling property (147) of the propagator implies for an n-point function folded with a test function $\varphi = \varphi(x_2, \ldots, x_n)$

$$\mathcal{L}^{\varepsilon,t}_{n,l}(x_1, \varphi; m^2, \lambda, g) = \rho^{4-n}\, \mathcal{L}^{\rho^2\varepsilon, \rho^2 t}_{n,l}(x_1, \varphi; \frac{m^2}{\rho^2}, \lambda, \rho^2 g)\ . \tag{150}$$

In the renormalization proof the CAS were constructed by imposing renormalization conditions for the relevant terms, see (67), (68), and by requiring the irrelevant terms to vanish at scale ε, see (59)-(61). As noted the renormalization conditions will now be supposed to be expressed in terms of intrinsic quantities, and they will be supposed to satisfy scaling (both statements are true for vanishing renormalization conditions). Because of the behaviour of $\sigma(x,y)^\mu$ under scaling, (145), this means

$$\begin{aligned}
a_l^{\varepsilon,\infty}(x; m^2, g) &= \rho^2\, a_l^{\rho^2\varepsilon,\infty}(x; \rho^{-2}\, m^2, \rho^2\, g) &(151)\\
f_l^{\mu,\varepsilon,\infty}(x; m^2, g) &= \rho^2\, f_l^{\mu,\rho^2\varepsilon,\infty}(x; \rho^{-2}\, m^2, \rho^2\, g) &(152)\\
b_l^{\mu\nu,\varepsilon,\infty}(x; m^2, g) &= \rho^2\, b_l^{\mu\nu,\rho^2\varepsilon,\infty}(x; \rho^{-2}\, m^2, \rho^2\, g) &(153)\\
c_l^{\varepsilon,\infty}(x; m^2, g) &= c_l^{\rho^2\varepsilon,\infty}(x; \rho^{-2}\, m^2, \rho^2\, g)\ . &(154)
\end{aligned}$$

For the standard case of ε-independent renormalization conditions the scaling of ε can of course be ignored. At the tree level the relation (150) holds as shown above. Using the FE with the standard inductive scheme it then follows that

(150) holds in the case of renormalization conditions satisfying (151)-(154).

Renormalization conditions imposed at some scale $t_R < \infty$ are in one to one relation to those imposed at $t = \infty$, and the local terms a_l^{ε,t_R} etc. can be viewed either as renormalization conditions imposed at this scale or as resulting from integrating the FE over $[t_R, \infty)$ with renormalization conditions imposed at ∞. From this fact and (150) one deduces that the relations corresponding to (151)-(154) for renormalization conditions imposed at finite t_R are

$$a_l^{\varepsilon,t_R}(x; m^2, g) = \rho^2\, a_l^{\rho^2\varepsilon,\rho^2 t_R}(x; \rho^{-2}\, m^2, \rho^2\, g) \quad \text{etc.} \tag{155}$$

In the subsequent analysis of the counter terms it will be helpful to first analyse the *massless* theory for t in the interval $[\varepsilon, T]$ to eliminate one of the parameters subject to scaling. While restricting to $[\varepsilon, T]$, the less singular corrections stemming from the massiveness (see (170) below) can be dealt with afterwards. The same can then be done

(trivially) for the finite contributions coming from integrating the FE of the massive theory over $[T,\infty)$.
For the massless theory we introduce the following notation : we denote

$$a_l^{\varepsilon,t}(x;g) \to a_{l,t_R}^{\varepsilon,t}(x;g)\,, \qquad \text{etc.}$$

to explicitly introduce all parameters subject to scaling, including the scale of the renormalization point t_R . Furthermore we will introduce the sequence of scales

$$t_n \; := \kappa^{-n}\, t_R, \quad \kappa > 1\,, \quad 1 \le n \le N\,, \;\; \text{such that} \;\; \varepsilon = t_N \;.$$

Then we use the shorthands

$$a_{l,t_R}^{n}(x;g) \; := a_{l,t_R}^{\varepsilon,t_n}(x;g) \qquad \text{etc.}\ , \tag{156}$$

and for the renormalization constants at $t = t_R$

$$a_l^{t_R}(x;g) \; := a_{l,t_R}^{\varepsilon,t_R}(x;g) \qquad \text{etc.} \tag{157}$$

As a consequence of the properties of the heat kernel, the terms $a_{l,t_R}^{n}(x;g)$ etc. are smooth scalars on the manifold. For the manifolds considered (of sectional curvature bounded above and below, as defined in Sect.2), we have proven bounds which are uniform in the curvature since our bounds on the heat kernel are uniform in this case. The same holds for their (low order) derivatives $(t\Delta)^s a_{l,t_R}^{n}(x;g)$ etc., since we obtain the same bounds for these derivatives due to (15). We can therefore decompose these terms according to their tensorial character into individual contributions from curvature, respecting the scaling property, such that in this decomposition there will only appear terms depending smoothly on the geometric quantities. This gives

$$a_{l,t_R}^{n}(x;g) = \alpha_{l,t_R}^{n} + R(x)\,\xi_{l,t_R}^{n} + \delta a_{l,t_R}^{n}(x;g) \tag{158}$$
$$f_{l,t_R}^{\mu,n}(x;g) = 0 + \delta f_{l,t_R}^{\mu,n}(x;g) \tag{159}$$
$$b_{l,t_R}^{\mu\nu,n}(x;g) = g^{\mu\nu}(x)\,\beta_{l,t_R}^{n} + \delta b_{l,t_R}^{\mu\nu,n}(x;g) \tag{160}$$
$$c_{l,t_R}^{n}(x;g) = \gamma_{l,t_R}^{n} + \delta c_{l,t_R}^{n}(x;g)\ . \tag{161}$$

The zero written in (159) reminds us that this term vanishes identically in the case of

constant curvature. The remainder terms in this decomposition may be analysed further

$$\delta a^{n}_{l,t_R}(x;g) = t_R\Big(\Delta R(x)\, h^{(1,n)}_l + R^2(x)\, h^{(1',n)}_l + R^{\mu\nu}(x) R_{\mu\nu}(x)\, h^{(1'',n)}_l + R^{\mu\nu\lambda\sigma}(x) R_{\mu\nu\lambda\sigma}(x)\, h^{(1''',n)}_l\Big) + \cdots \tag{162}$$

$$\delta f^{\mu,\,n}_{l,t_R}(x;g) = t_R\, g^{\mu\nu}(x)\, R_{,\nu}(x)\, h^{(2,n)}_l + \cdots \tag{163}$$

$$\delta b^{\mu\nu,\,n}_{l,t_R}(x;g) = t_R\Big(R^{\mu\nu}(x)\, h^{(3,n)}_l + g^{\mu\nu}(x) R(x) h^{(3',n)}_l\Big) + \cdots \tag{164}$$

$$\delta c^{n}_{l,t_R}(x;g) = t_R\, R(x)\, h^{(4,n)}_l + \cdots\ . \tag{165}$$

All the h-functions in this decomposition have mass dimension zero and are therefore independent of t_R which is the only scale. The dots indicate terms of higher scaling dimension in the expansion w.r.t. curvature terms. We then
assume that these expansions are asymptotic [21]*, in the sense that the remainders satisfy*

$$|\,\delta a^{n}_{l,t_R}(x;\rho^2\, g)|\,,\ |\,\omega_\mu(x)\,\delta f^{\mu,\,n}_{l,t_R}(x;\rho^2 g)|\,,\ |\,\omega^{(2)}_{\mu\nu}(x)\,\delta b^{\mu\nu,\,n}_{l,t_R}(x;\rho^2 g)|\ \le\ O(\rho^{-4})\ , \tag{166}$$

$$|\,\delta c^{n}_{l,t_R}(x;\rho^2 g)|\ \le\ O(\rho^{-2})\ . \tag{167}$$

Here n and t_R are (of course) kept fixed and furthermore, the rank 1 resp. rank 2 cotensor fields $\omega_\mu(x)$, $\omega^{(2)}_{\mu\nu}(x)$ are assumed to stay invariant under scaling $g \to \rho^2\, g$. The bounds are in agreement with the leading terms written in (162)-(165). This assumption appears plausible and is often taken for granted, see e.g. [HoWa3]. Its proof requires a more thorough analysis of the heat kernel and its convolutions than is given here.

Proposition 4 :
Assuming (166),(167), then for position independent coupling λ there exist renormalization conditions of the form (67, 68) such that the bare action takes the simple form (21), this means that for $l \ge 1$

$$L^\varepsilon_l(\varphi) = \frac{1}{2}\int_x \{(\,\alpha^\varepsilon_l + \xi^\varepsilon_l\ R(x))\,\varphi^2(x)\ -\ b^\varepsilon_l\,\varphi(x)\Delta\varphi(x)\ +\ \frac{2}{4!}\,c^\varepsilon_l\,\varphi^4(x)\} \tag{168}$$

with the following bounds

$$|\,\alpha^\varepsilon_l\,| \le\ \frac{1}{\varepsilon}\,\mathcal{P}_{l-1}\log\frac{1}{\varepsilon}\,,\quad |\,\xi^\varepsilon_l\,| \le\ \mathcal{P}_l\log\frac{1}{\varepsilon}\,,\quad |\,b^\varepsilon_l\,| \le\ \mathcal{P}_{l-1}\log\frac{1}{\varepsilon}\,,\quad |\,c^\varepsilon_l\,| \le\ \mathcal{P}_l\log\frac{1}{\varepsilon}\,. \tag{169}$$

Proof :
We first note that Proposition 1 can be proven in complete analogy when imposing renormalization conditions of the form (67), (68) at scale $t_R = T > 0$ for the massless theory.

[21] asymptoticity is obviously required up to second order in ρ^2 only.

The scale T is the one up to which we have precise control on the heat kernel, cf. (13), and it is thus related to the geometry of $\mathcal{M}$. Furthermore we can expand for $\varepsilon \le t \le T$

$$\mathcal{L}^{\varepsilon,t}_{n,l}(m^2;x_1,\varphi_{\tau,y_{2,s}}) \;=\; \mathcal{L}^{\varepsilon,t}_{n,l}(0;x_1,\varphi_{\tau,y_{2,s}}) \;+\; m^2\,\partial_{m^2}\mathcal{L}^{\varepsilon,t}_{n,l}(0;x_1,\varphi_{\tau,y_{2,s}}) \qquad (170)$$

$$+\; m^4 \int_0^1 d\lambda\;(1-\lambda)\,\partial^2_{m^2}\mathcal{L}^{\varepsilon,t}_{n,l}(\lambda m^2;x_1,\varphi_{\tau,y_{2,s}})\;.$$

We first analyse the massless theory and then comment on the derivative terms.

We use the notation (156), (157). The theory is specified through renormalization conditions of the form (67), (68) imposed at scale $t_R = T$:

$$a^T_l(x;g)=0\,,\;\; f^{\mu,T}_l(x;g)=0\,,\;\; b^{\mu\nu,T}_l(x;g)=0\,,\;\; c^T_l(x;g)=0\;, \qquad (171)$$

together with boundary conditions of the type (59)-(61) at scale $\varepsilon=\kappa^{-N}T$ for $l'\le l$. Our aim is to analyse the bare action. From Proposition 1 we obtain for $l>0$ the bounds

$$|\,a^n_{l,T}(x;g)| \le O(1)\;\kappa^n\; n^{l-1} \qquad (172)$$

$$|\,f^{\mu,\,n}_{l,T}(x;g)\,\omega_\mu(x)| \le O(1)\;|\,\omega(x)|\;\kappa^{\frac{n}{2}}\; n^{l-1} \qquad (173)$$

$$|\,b^{\mu\nu,\,n}_{l,T}(x;g)\,\omega^{(2)}_{\mu\nu}(x)| \le O(1)\;|\,\omega^{(2)}(x)|\; n^{l-1} \qquad (174)$$

$$|\,c^n_{l,T}(x;g)| \le O(1)\; n^l\,. \qquad (175)$$

In the sequel we present the detailed argument for the relevant term $a(x;g)$, whereas the analogous treatment of the other ones is stated in shortened form. In view of the decomposition (158) we want to prove inductively in n [22]

$$|\alpha^n_{l,T}| \le O(1)\sum_{n'=1}^{n}\kappa^{n'}\;{n'}^{\,l-1}\,,\;\; |\xi^n_{l,T}| \le O(1)\sum_{n'=1}^{n}{n'}^{\,l-1}\,,\;\; |\delta a^n_{l,T}(x;g)| \le O(1)\sum_{n'=1}^{n}\kappa^{-n'}{n'}^{l-1}\;. \qquad (176)$$

First note that the uniqueness of the solutions of the FE implies that the relevant term $a^{n+1}_{l,T}(x,g)$ satisfies

$$a^{n+1}_{l,T}(x;g) = \hat a^1_{l,\kappa^{-n}T}(x;g) \qquad (177)$$

where $\hat a^1_{l,\kappa^{-n}T}(x;g)$ is defined to be the corresponding relevant term at scale $\kappa^{-(n+1)}T$ for the theory renormalized at scale $\kappa^{-n}T$, with renormalization conditions of the following form

$$\hat a^{\kappa^{-n}T}_l(x;g) = a^n_{l,T}(x;g) \quad \text{(analogously for the } f\,,\ b\,,\ c\text{-terms)}\;. \qquad (178)$$

[22]More precisely induction is in (l,n) in the order $(l,1),(l,2),\ldots,(l,N),(l+1,1),\ldots$, but the step $(l,N)\to(l+1,1)$ is trivial.

This just means that we take renormalization conditions at scale T, integrate down to $\kappa^{-n}T$, and take the values we arrive at for the local terms, as renormalization conditions at the scale $\kappa^{-n}T$. By the uniqueness statement we obtain the same Schwinger functions as when imposing $a_l^T(x;g)$ etc. at scale T.
From the scaling relations, cf. (155), we have

$$a_{l,T}^{n}(x;g) \;=\; \kappa^n\; a_{l,\kappa^n T}^{n}(x;\kappa^n g) \;=\; \kappa^n\; \hat{a}_l^{T}(x;\kappa^n g)\,, \tag{179}$$

$$a_{l,T}^{n+1}(x;g) \;=\; \kappa^n\; a_{l,\kappa^n T}^{n+1}(x;\kappa^n g) \;=\; \kappa^n\; \hat{a}_{l,T}^{1}(x;\kappa^n g)\;. \tag{180}$$

In the case of $c_{l,T}^{n}(x;g)$ such relations hold without the external factor κ^n. Moreover,

$$b_{l,T}^{\mu\nu,\,n}(x;g)\,\omega^{(2)}_{\mu\nu}(x) \;=\kappa^n\; \hat{b}_l^{\mu\nu,T}(x;\kappa^n g)\,\omega^{(2)}_{\mu\nu}(x)\,, \tag{181}$$
$$b_{l,T}^{\mu\nu,\,n+1}(x;g)\,\omega^{(2)}_{\mu\nu}(x) \;=\kappa^n\; \hat{b}_{l,T}^{\mu\nu,1}(x;\kappa^n g)\,\omega^{(2)}_{\mu\nu}(x)\;, \tag{182}$$

and the analogue for f^μ is obtained replacing $b^{\mu\nu}$ by f^μ and $\omega^{(2)}_{\mu\nu}(x)$ by $\omega_\mu(x)$. Using (172)-(175) and (179)-(182), we then obtain

$$|\,\hat{a}_l^{T}(x;\kappa^n g)| \;\le\; O(1)\;n^{l-1}\,, \quad |\,\hat{f}_l^{\mu,T}(x;\kappa^n g)\,\omega_\mu(x)| \;\le\; O(1)\;|\,\omega(x)|_{\kappa^n g}\;n^{l-1}\,, \tag{183}$$

$$|\,\hat{b}_l^{\mu\nu,T}(x;\kappa^n g)\,\omega^{(2)}_{\mu\nu}(x)| \;\le\; O(1)\;|\,\omega^{(2)}(x)|_{\kappa^n g}\;n^{l-1}\,, \quad |\,\hat{c}_l^{T}(x;\kappa^n g)| \;\le\; O(1)\;n^{l} \tag{184}$$

where we denoted by $|\cdot|_{\kappa^n g}$ the norm (A.32) generated by $\kappa^n g$.

We now consider more general massless Schwinger functions $\hat{\mathcal{L}}_{p,l}^{\kappa^{-1}T,t}(x_1,\varphi_{\tau,y_{2,s}};\tilde{g})$ resulting from a metric $\tilde{g}$ of the class defined in Section 2 [23] and satisfying renormalization conditions of the form (183), (184) at loop orders $l' < l$. At loop order l we first assume vanishing renormalization conditions. Afterwards the contribution coming from renormalization conditions at loop order l, bounded as in (183),(184) will be added to the result obtained. Integrating the flow equations for these Schwinger functions within the interval $[\kappa^{-1}T,\,T]$, one verifies with the aid of the usual inductive scheme and analogously as in Proposition 1, for $t \in [\kappa^{-1}T,\,T]$, the bounds

$$|\,\hat{\mathcal{L}}_{p,l}^{\kappa^{-1}T,t}(x_1,\varphi_{\tau,y_{2,s}};\tilde{g})| \;\le\; O(1)\;n^{l}\;\mathcal{F}_{s,l}(t,\tau)\;, \quad p \ge 6 \tag{185}$$

$$|\,\hat{\mathcal{L}}_{p,l}^{\kappa^{-1}T,t}(x_1,\varphi_{\tau,y_{2,s}};\tilde{g})| \;\le\; O(1)\;n^{l-1}\;\mathcal{F}_{s,l}(t,\tau)\;, \quad p \le 4\;. \tag{186}$$

These bounds are dictated by the size of the boundary conditions for $l' < l$ which enter on the r.h.s. of the FE. In fact one realizes that the factors of $n^{l'}$ appearing in the bound on the r.h.s. can be factored out and majorized by n^l resp. n^{l-1}. The remainder is then

[23] $\tilde{g} = \kappa^n g$ certainly belongs to this class if g does

inductively bounded (uniformly in n) by the $\mathcal{F}_{s,l}(t,\tau)$-factors times a (p,l)-dependent constant. For the relevant terms these bounds imply

$$
\begin{aligned}
&|\,\hat{a}^{1,0}_{l,T}(x;\tilde{g})| \;\leq\; O(1)\;n^{l-1}\,, \quad |\,\hat{f}^{\mu,1,0}_{l,T}(x;\tilde{g})\,\omega_\mu(x)| \;\leq\; O(1)\;|\,\omega(x)|_{\tilde{g}}\;n^{l-1}\,, \\
&|\,\hat{b}^{\mu\nu,1,0}_{l,T}(x;\tilde{g})\,\omega^{(2)}_{\mu\nu}(x)| \;\leq\; O(1)\;|\,\omega^{(2)}(x)|_{\tilde{g}}\;n^{l-1}\,, \quad |\,\hat{c}^{1,0}_{l,T}(x;\tilde{g})| \;\leq\; O(1)\;n^{l-1}\;.
\end{aligned}
\tag{187}
$$

Here the upper index 0 indicates that we were calculating with vanishing renormalization conditions at loop order l. On decomposing as in (158)

$$
\hat{a}^{1,0}_{l,T}(x;\tilde{g}) = \hat{\alpha}^{1,0}_{l,T} + \tilde{R}(x)\,\hat{\xi}^{1,0}_{l,T} + \delta\hat{a}^{1,0}_{l,T}(x;\tilde{g}) \tag{188}
$$

we then obtain from (187) by linear independence

$$
|\,\hat{\alpha}^{1,0}_{l,T}\,|\,,\;|\,\hat{\xi}^{1,0}_{l,T}\,|\,,\;|\,\delta\hat{a}^{1,0}_{l,T}(x;\tilde{g})\,|\;\leq O(1)\;n^{l-1}\;. \tag{189}
$$

Specializing to $\tilde{g} = \kappa^n g$ in (188) yields

$$
\hat{a}^{1,0}_{l,T}(x;\kappa^n g) = \hat{\alpha}^{1,0}_{l,T} + \kappa^{-n}R(x)\,\hat{\xi}^{1,0}_{l,T} + \delta\hat{a}^{1,0}_{l,T}(x;\kappa^n g) \tag{190}
$$

where our smoothness assumption (166) on $\delta\hat{a}^{1,0}_{l,T}(x;\kappa^n g)$ implies

$$
|\,\delta\hat{a}^{1,0}_{l,T}(x;\kappa^n g)| \;\leq\; O(1)\;\kappa^{-2n}\;n^{l-1}\;. \tag{191}
$$

Upon scaling according to (180) we then obtain

$$
a^{n+1,0}_{l,T}(x;g) = \alpha^{n+1,0}_{l,T} + R(x)\,\xi^{n+1,0}_{l,T} + \delta a^{n+1,0}_{l,T}(x;g) \tag{192}
$$

with the bounds

$$
|\alpha^{n+1,0}_{l,T}| \leq O(1)\;\kappa^n\;n^{l-1}\,,\quad |\xi^{n+1,0}_{l,T}| \leq O(1)\;n^{l-1}\,,\quad |\delta a^{n+1,0}_{l,T}(x;g)| \leq O(1)\,\frac{n^{l-1}}{\kappa^n}\;. \tag{193}
$$

Adding the contributions from the renormalization condition obeying the inductive bounds from (176), then yields

$$
\begin{aligned}
|\,\alpha^{n+1}_{l,T}\,| &\leq O(1)\sum_{n'=1}^{n+1}\kappa^{n'}{n'}^{\,l-1} \leq\; O(1)\;\kappa^{n+1}(n+1)^{\,l-1}\;, \\
|\,\xi^{n+1}_{l,T}\,| &\leq O(1)\sum_{n'=1}^{n+1}{n'}^{\,l-1} \leq\; O(1)\,(n+1)^l\;, \\
|\,\delta a^{n+1}_{l,T}(x;g)\,| &\leq\; O(1)\sum_{n'=1}^{n+1}\kappa^{-n'}{n'}^{\,l-1} \leq O(1)\;,
\end{aligned}
$$

thus establishing the bounds (176) by induction. The statement for $n+1=N$ implies Proposition 4, noting in particular that the last inequality allows for eliminating the term $\delta a^N_{l,T}(x;g)$ by a *finite* change of the corresponding renormalization condition at scale T.

The other relevant terms are dealt with analogously. Regarding c_l we obtain in place of (192), (193)

$$c^{n+1,0}_{l,T}(x;g) = \gamma^{n+1,0}_{l,T} + \delta c^{n+1,0}_{l,T}(x;g)\,, \tag{194}$$

$$|\gamma^{n+1,0}_{l,T}| \leq O(1)\; n^{l-1}\,, \quad |\delta c^{n+1,0}_{l,T}(x;g)| \leq O(1)\,\kappa^{-n}\, n^{l-1}\;. \tag{195}$$

As for $b^{\mu\nu}_l$, decomposing as in (160)

$$\hat{b}^{\mu\nu,\,1,0}_{l,T}(x;\tilde{g}) = \;\tilde{g}^{\,\mu\nu}(x)\,\hat{\beta}^{1,0}_{l,T} + \delta\hat{b}^{\mu\nu,\,1,0}_{l,T}(x;\tilde{g}) \tag{196}$$

we get from (187)

$$|\,\tilde{g}^{\,\mu\nu}(x)\,\omega^{(2)}_{\mu\nu}(x)\,\hat{\beta}^{1,0}_{l,T}|\,,\;\; |\,\omega^{(2)}_{\mu\nu}(x)\,\delta\hat{b}^{\mu\nu,\,1,0}_{l,T}(x;\tilde{g})| \;\leq\; O(1)\;|\,\omega^{\,(2)}(x)|_{\tilde{g}}\; n^{l-1}\;. \tag{197}$$

The second bound implies for $\tilde{g}\,=\,g$

$$|\,\omega^{(2)}_{\mu\nu}(x)\,\delta\hat{b}^{\mu\nu,\,1,0}_{l,T}(x;g)| \;\leq\; O(1)\;|\,\omega^{\,(2)}(x)|_{g}\; n^{l-1}\;,$$

and using (166) then provides

$$|\,\omega^{(2)}_{\mu\nu}(x)\,\delta\hat{b}^{\mu\nu,\,1,0}_{l,T}(x;\kappa^n g)| \;\leq\; O(1)\;|\,\omega^{\,(2)}(x)|_{g}\;\kappa^{-2n}\; n^{l-1}\;. \tag{198}$$

Upon scaling, (182), and observing $\kappa^n\,|\,\omega^{\,(2)}(x)|_{\,\tilde{g}} = |\,\omega^{\,(2)}(x)|_g$, we obtain from (196)-(198)

$$|\,g^{\,\mu\nu}(x)\,\omega^{(2)}_{\mu\nu}(x)\,\beta^{n+1,0}_{l,T}| \;\;\leq\;\; O(1)\;|\,\omega^{\,(2)}(x)|_{g}\; n^{l-1}\,, \tag{199}$$

$$|\,\omega^{(2)}_{\mu\nu}(x)\,\delta b^{\mu\nu,\,n+1,0}_{l,T}(x;g)| \;\leq\; O(1)\;|\,\omega^{\,(2)}(x)|_{g}\;\kappa^{-n}\, n^{l-1}\;. \tag{200}$$

Finally, proceeding similarly we find

$$|\,\omega_\mu(x)\,\delta f^{\mu,\,n+1,0}_{l,T}(x;g)| \;\leq\; O(1)\;|\omega(x)|_g\;\kappa^{-n}\, n^{l-1}\,. \tag{201}$$

Since (200), (201) hold with general $\omega^{\,(2)}$ and ω, respectively, the bounds extend to the individual tensorial components.

The proof of Proposition 4 is finished through the following remarks :

i) To go back to the massive theory we have to add the two derivative terms from (170). An m^2-derivative acting on the propagator produces an additional factor of t. As a consequence of this we get the bounds

$$|\partial^s_{m^2}\mathcal{L}^{\varepsilon,t}_{n,l}(x_1,\varphi_s)| \;\leq\; t^{\frac{n+2s-4}{2}}\;\mathcal{P}_l\log(t,\tau)^{-1}\;\mathcal{F}_{s,l}(t,\tau)\;. \tag{202}$$

This implies that for $s \geq 1$ there is only one relevant term

$$\int_{x_2} \partial_{m^2} \mathcal{L}^{\varepsilon,t}_{2,l}(x_1, x_2)$$

which by the previous statement is logarithmically bounded. Applying the expansion (158) to this term, all terms produced can be absorbed -respecting the bounds- in the terms already present in the massless theory. So the previous result is maintained.

ii) We restore the massive theory at scale T by adding the contributions from the last two terms on the r.h.s. of (170). According to Proposition 3, renormalization conditions at scale T can then be translated into renormalization conditions at scale $t \to \infty$ for the massive theory. ∎

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A Some Notions from Riemannian Geometry

Here, we briefly recall some basic properties of Riemannian manifolds pertinent to the main text and thereby introduce the definitions and conventions used. For a detailed exposition we refer to [Wil]. We consider a connected four-dimensional smooth manifold $\mathcal{M}$. A Riemannian metric on $\mathcal{M}$ is a tensor field g of type (0,2) (more technically: a section of $\otimes^2 \mathcal{T}^*\mathcal{M}$, where $\mathcal{T}^*\mathcal{M}$ is the cotangent bundle) which associates to each point $p \in \mathcal{M}$ a positive-definite inner product on $\mathcal{T}_p\mathcal{M}$, the tangent space to $\mathcal{M}$ at p. Given a chart with local coordinates $x = (x^1, x^2, x^3, x^4) \in \mathbf{R}^4$, and denoting by $\partial_\mu := \partial/\partial x^\mu$ and by $dx^\mu, \mu = 1, 2, 3, 4$, the corresponding coordinate vector and covector fields, respectively, the Riemannian metric tensor has the form

$$g \;=\; g_{\mu\nu}(x)\, dx^\mu \otimes dx^\nu\,, \qquad g_{\mu\nu}(x) \;=\; g(\partial_\mu, \partial_\nu). \tag{A.1}$$

At each point x the components $g_{\mu\nu}(x)$ form the entries of a symmetric positive-definite matrix. In (A.1) and henceforth the summation convention is implied. Moreover, with

$$g^{\lambda\mu}(x) g_{\mu\nu}(x) := \delta^\lambda_\nu\,, \qquad |g(x)| \equiv \det\bigl(g_{\mu\nu}(x)\bigr) \tag{A.2}$$

the Riemannian volume element reads

$$dV(x) \;=\; |g(x)|^{\frac{1}{2}}\, dx^1 dx^2 dx^3 dx^4\,, \tag{A.3}$$

and the Laplace-Beltrami operator acting on a scalar field is defined by

$$\Delta\phi(x) \;=\; |g(x)|^{-\frac{1}{2}}\partial_\mu\, g^{\mu\nu}(x)|g(x)|^{\frac{1}{2}}\,\partial_\nu\,\phi(x)\,. \tag{A.4}$$

The Levi-Civita connection ∇ of the Riemannian metric g leads to the covariant derivative of the coordinate vector fields

$$\nabla_{\partial_\nu}\partial_\mu \;=\; \Gamma^\lambda_{\mu\nu}(x)\,\partial_\lambda \tag{A.5}$$

with the Christoffel symbols

$$\Gamma^\lambda_{\mu\nu}(x) \;=\; \frac{1}{2}g^{\lambda\varrho}\big(\partial_\mu\, g_{\varrho\nu} + \partial_\nu\, g_{\varrho\mu} - \partial_\varrho\, g_{\mu\nu}\big) \;=\; \Gamma^\lambda_{\nu\mu}\,. \tag{A.6}$$

The Riemannian curvature tensor R of the connection ∇ maps the triple of vector fields X, Y, Z to the vector field [24]

$$R(X,Y)Z \;=\; \big(\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}\big)Z\,. \tag{A.7}$$

In local coordinates with $X = X^\mu(x)\,\partial_\mu$ and similarly for Y, Z the curvature tensor has the form

$$R(X,Y)Z \;=\; R^\varrho{}_{\sigma\mu\,\nu}\,Z^\sigma X^\mu\, Y^\nu\,\partial_\varrho \tag{A.8}$$

with components

$$R^\varrho{}_{\sigma\mu\,\nu}(x) \;=\; \partial_\mu\,\Gamma^\varrho_{\sigma\nu} - \,\partial_\nu\,\Gamma^\varrho_{\sigma\mu} + \Gamma^\varrho_{\lambda\mu}\,\Gamma^\lambda_{\sigma\nu} - \Gamma^\varrho_{\lambda\nu}\,\Gamma^\lambda_{\sigma\mu}\,. \tag{A.9}$$

The components of the Ricci tensor follow by internal contraction as

$$R_{\sigma\nu}(x) \;:=\; R^\mu{}_{\sigma\mu\nu}(x)\,, \tag{A.10}$$

and the Ricci curvature at the point p with local coordinates x in the direction of the tangent vector $v \in \mathcal{T}_p\mathcal{M}$ is defined by

$$Ric_p(v) \;:=\; \frac{R_{\sigma\nu}(x)\,v^\sigma v^\nu}{g_{\sigma\nu}(x)\,v^\sigma v^\nu}\,. \tag{A.11}$$

Moreover, the *scalar curvature* is given by

$$R(x) \;:=\; g^{\sigma\nu}(x)\,R_{\sigma\nu}(x)\,. \tag{A.12}$$

Let $v, w \in \mathcal{T}_p\,\mathcal{M}$ span the two-dimensional subspace S. Then the *sectional curvature* of $\mathcal{M}$ at the point p along the section S is defined as

$$Sec_p(v,w) := -\,\frac{g_p(R_p(v,w)v,w)}{g_p(v,v)g_p(w,w) - g_p(v,w)^{\,2}}\,. \tag{A.13}$$

[24]There is obviously a freedom in choosing an overall sign, which has to be observed, similarly in the case of the Ricci tensor.

It depends only on the section S, not on the spanning vectors v, w. Given in $\mathcal{T}_p\mathcal{M}$ an orthonormal basis $\xi_{(r)}, r = 1, .., 4$, with components $\{\xi^\mu_{(r)}\}$ implies

$$g^{\mu\nu}(x) \;=\; \sum_{r=1}^{4} \xi^\mu_{(r)}\,\xi^\nu_{(r)} \tag{A.14}$$

and leads to sectional curvatures, $r \neq s$,

$$Sec_p\,(\xi_{(r)}, \xi_{(s)}) = R_{\sigma\alpha\mu\nu}(x)\,\xi^\sigma_{(r)}\,\xi^\alpha_{(s)}\,\xi^\mu_{(r)}\,\xi^\nu_{(s)}\,. \tag{A.15}$$

Herefrom it follows that

$$Ric_p(\xi_{(s)}) \;=\; \sum_{r,\,r\neq s} Sec_p\,(\xi_{(r)}, \xi_{(s)})\,, \tag{A.16}$$

$$R(x) \;=\; 2\sum_{r<\,s} Sec_p\,(\xi_{(r)}, \xi_{(s)})\,. \tag{A.17}$$

The geodesics passing through a point $p \in \mathcal{M}$ can in general only be defined for values of the (affine) parameter confined to a finite interval. They generate a map from an open domain of the tangent space into the manifold, called the exponential map, $\exp : \Omega \subset \mathcal{T}_p\mathcal{M} \to \mathcal{M}$. Its inverse are the Riemannian normal coordinates. A manifold is geodesically complete, if this parameter interval everywhere extends to $\mathbf{R}$, and hence $\Omega = \mathcal{T}_p\mathcal{M}$, for all $p \in \mathcal{M}$. For points $p, q \in \mathcal{M}$ the distance function $d(p, q) = d(q, p)$ is defined by $d(p, q) \;=\; \inf_\alpha L(\alpha)$, where α runs over all C^1 curve segments joining p to q, i.e. $\alpha : [a, b\,] \to \mathcal{M}$, $\alpha(a) = p$, $\alpha(b) = q$, and its arc length given by

$$L(\alpha) \;=\; \int_a^b dt\,\Big(\,g_{\alpha(t)}\big(\dot\alpha(t), \dot\alpha(t)\big)\Big)^{\frac{1}{2}}\,. \tag{A.18}$$

If p is sufficiently close to q there is always a unique geodesic determining $d(p, q)$. Regarding a geodesic ball in $\mathcal{M}$ with center p and with radius r ,

$$\mathcal{B}(p, r) = \{q \in \mathcal{M}|\, d(p, q) < r\}\,, \tag{A.19}$$

its Riemannian volume is denoted by

$$|\mathcal{B}(p, r)| \;=\; \int_{\mathcal{B}} dV\,. \tag{A.20}$$

For $x, y \in \mathcal{M}$ we introduce the bi-tensor of type scalar-vector

$$\sigma(x, y)^\mu \;:=\; \frac{1}{2} g^{\mu\nu}(y)\,\frac{\partial}{\partial y^\nu}\;d^{\,2}(x, y) \tag{A.21}$$

which satisfies

$$\sigma(x,y)^{\mu}\ \sigma(x,y)^{\nu}\ g_{\mu\nu}(y) \ = \ d^2(x,y)\ . \tag{A.22}$$

In the renormalization proof we need covariant Taylor expansion formulae in the Schlömilch form, i.e. with integrated remainders, which are obtained as follows: [25] Given a complete Riemannian manifold $(\mathcal{M},g)$, and a chart $(\mathcal{U},x)$ with local coordinates x, a geodesic $x(s)$ parametrized by its arc length s satisfies

$$\ddot{x}^{\lambda}(s)+\Gamma^{\lambda}_{\mu\nu}(x(s))\,\dot{x}^{\mu}(s)\,\dot{x}^{\nu}(s) \ = \ 0\,, \tag{A.23}$$

$$g_{\mu\nu}(x(s))\,\dot{x}^{\mu}(s)\,\dot{x}^{\nu}(s) \ = \ 1\,. \tag{A.24}$$

Let $f\in C^{\infty}(\mathcal{M})$ and $F(s):=f(x(s))$, then

$$\Big(\frac{d}{ds}\Big)^n F(s) = \big(\nabla_{\nu_n}\cdots\nabla_{\nu_1}f\big)(x(s))\,\dot{x}^{\nu_1}(s)\cdots\dot{x}^{\nu_n}(s)\,. \tag{A.25}$$

The proof is by induction, using (A.23).

We consider the geodesic segment with initial point $x_0=x(0)$ and end point $x=x(s)$, hence $d(x,x_0)=s$. With (A.21) we then have the relation, see e.g. [Wil, sect. 6.3],

$$\sigma(x,x_0)^{\nu} \ = \ -s\,\dot{x}^{\nu}(0)\,. \tag{A.26}$$

From the Taylor formula with remainder

$$F(s)=F(0)+\sum_{l=1}^{n}\frac{s^l}{l!}\,F^{(l)}(0)+R_n\,,\qquad R_n=\int_0^s dr\,\frac{(s-r)^n}{n!}\,F^{(n+1)}(r)\,, \tag{A.27}$$

we obtain, using (A.25), (A.26),

$$\begin{aligned} f(x)=f(x_0) \ &+ \ \sum_{l=1}^{n}\frac{(-1)^l}{l!}\ \sigma(x,x_0)^{\nu_l}\cdots\sigma(x,x_0)^{\nu_1}\big(\nabla_{\nu_l}\cdots\nabla_{\nu_1}f\big)(x_0)\\ &+ \ R_n\,, \end{aligned} \tag{A.28}$$

$$R_n(x,x_0)=\int_0^{d(x,x_0)} dr\,\frac{(d(x,x_0)-r)^n}{n!}\ \dot{x}^{\nu_{n+1}}(r)\cdots\dot{x}^{\nu_1}(r)\big(\nabla_{\nu_{n+1}}\cdots\nabla_{\nu_1}f\big)(x(r))\,. \tag{A.29}$$

Between fixed x,x_0 we can reparametrize the geodesic segment $x(r)=X(\rho)$, with $r=d(x_0,x)\rho$, $0\leq\rho\leq 1$, implying $g_{\mu\nu}(X(\rho))\,\dot{X}^{\mu}(\rho)\dot{X}^{\nu}(\rho)=d^2(x_0,x)$. Then

$$R_n(x,x_0)=\int_0^1 d\rho\,\frac{(1-\rho)^n}{n!}\ \dot{X}^{\nu_{n+1}}(\rho)\cdots\dot{X}^{\nu_1}(\rho)\big(\nabla_{\nu_{n+1}}\cdots\nabla_{\nu_1}f\big)(X(\rho))\,. \tag{A.30}$$

[25] We give the complete argument since we only found part of it in the literature [BaVi].

In the remainder R_n the contraction of a tensor of type $(n+1,0)$ with a tensor of type $(0,n+1)$ can be viewed via the (inverse) Riemannian metric as the scalar product of two tensors of type $(0,n+1)$. To bound $|R_n(x,x_0)|$, Cauchy's inequality is used observing (A.24),

$$|R_n(x,x_0)| \leq \tag{A.31}$$

$$\int_0^{d(x,x_0)} dr\, \frac{(d(x,x_0)-r)^n}{n!}\; |(\nabla^{n+1}f)(x(r))| = d^{\,n+1}(x_0,x) \int_0^1 d\rho\, \frac{(1-\rho)^n}{n!}\; |(\nabla^{n+1}f)(X(\rho))| \, ,$$

where the norm square is given by

$$|(\nabla^{n+1}f)(x)|^2 =$$
$$\big(\nabla_{\mu_{n+1}}\cdots\nabla_{\mu_1}f\big)(x)\, g^{\mu_{n+1}\nu_{n+1}}(x)\cdots g^{\mu_1\nu_1}(x)\big(\nabla_{\nu_{n+1}}\cdots\nabla_{\nu_1}f\big)(x)\,. \tag{A.32}$$

Majorising in (A.31) the norm on the geodesic segment γ between x_0 and x yields the bound

$$|\, R_n(x,x_0)| \leq \frac{d^{\,n+1}(x,x_0)}{(n+1)!}\; \sup_{y\in\gamma} |(\nabla^{n+1}f)(y)|\,. \tag{A.33}$$

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